Week 5 and 6 Notes

Contunity

If a function is continuous, it must pass these 3 rules

1. c is in the domain of f(x)
2. f(c) exists
3. $\lim_{x \to c} f(x) = f(c)$

Eg. Write a delta-epsilon proof to show that $f(x) = 2x+1$ is continuous at $x=5$.

1. Dom of $f(x) = (-\infty, \infty)$

   $5 \in \text{Dom } f(x)$

2. $f(5) = 2(5)+1$

   $= 11$

3. Prove $\lim_{x \to 5} 2x+1 = 11$

$\forall \varepsilon > 0, \exists \delta > 0 \mid 0 < |x-5| < \delta \rightarrow |2x-10| < \varepsilon$

$|2x-10| < \varepsilon$

$2|x-5| < \varepsilon$

$2\delta = \varepsilon$

$\delta = \varepsilon/2$

$|2x-10| < \varepsilon$

$2|x-5|$

$= 2\delta$

$= 2(\varepsilon/2)$

$= \varepsilon$

QED

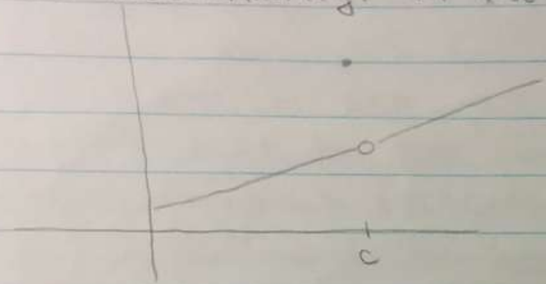
$\therefore \lim_{x \to 5} 2x+1 = 11$

$\therefore f(x)$ is continuous at $x=5$

Types of Discontinuities

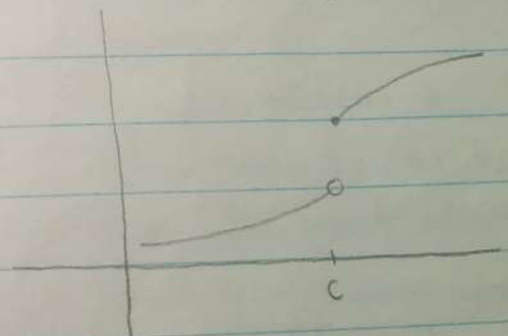
1. Removable Discontinuity: If $\lim_{x \to c} f(x)$ exists but isn't equal to $f(c)$

E.g.



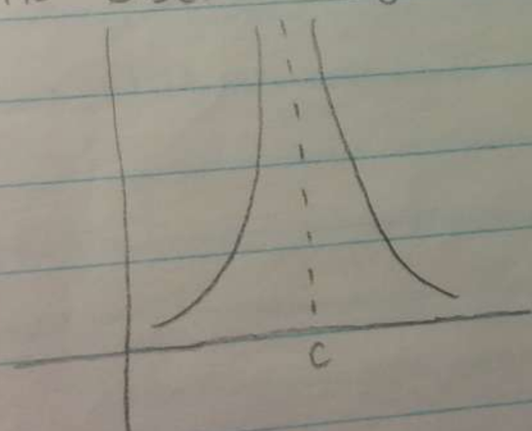
2. Jump Discontinuity: If $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist but are not equal.

E.g.



3. Infinite Discontinuity: If one or both of $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ is infinite.

E.g.



Continunity at a point:
If function f is defined on open interval $(c-p, c+p)$, we say that f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.

The precise definition of contunity at a point is:
If function f is defined on open interval $(c-p, c+p)$, we say that f is continuous at c if for each $\varepsilon > 0$, there exists $d > 0$ such that if $|x-c| < d$ then $|f(x) - f(c)| < \varepsilon$.

One Sided Limits

- $f(x)$ is cont from the right at number c if $\lim_{x \to c^+} f(x) = f(c)$.

$\forall \varepsilon > 0, \exists d > 0 \mid |x \in (c, c+d)| \to |f(x) - f(c)| < \varepsilon$

- $f(x)$ is cont from the left at number c if $\lim_{x \to c^-} f(x) = f(c)$

$\forall \varepsilon > 0, \exists d > 0 \mid |x \in (c-d, c)| \to |f(x) - f(c)| < \varepsilon$

Contunity on an Interval

- Function f is cont on open interval (a,b) if it is cont at every number in the interval

For all $p \in (a,b)$: $\lim_{x \to p} f(x) = f(p)$

- Function f is cont on closed interval $[a,b]$ if it is cont on open interval (a,b), right cont at a and left cont at b.

For all $p \in (a,b)$: $\lim_{x \to p} f(x) = f(p)$

$\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$

3 Important Consequences of Contunity

1. Cont functions on closed interval [a,b] attain their min and max values on that interval.

2. Values of cont functions on closed interval go through every possible value between f(a) and f(b).

3. Cont functions on closed interval are bounded and attain their upper and lower bounds.

- Extreme Value Theorem:

If $f(x)$ is cont on closed interval $[a,b]$, then there exists some values M and m in $[a,b]$ such that $f(M)$ is the max value and $f(m)$ is the min value.

Proof:

Lemma

If f is cont on $[a,b]$, then f is bounded on $[a,b]$.

Proof

Consider $\{x : x \in [a,b]$ and f is bounded on $[a,x]\}$

It is easy to see that this set is non-empty and bounded above by b. Thus, we can set $c = \text{lub}\{x : f$ is bounded on $[a,x]\}$

To argue $c=b$, suppose $c<b$. Since f is cont at c, it is bounded on $[c-\varepsilon, c+\varepsilon]$ for some $\varepsilon>0$. This means that f is bounded on $[a, c-\varepsilon]$ and on $[c-\varepsilon, c+\varepsilon]$, which means f is bounded on $[a, c+\varepsilon]$. However, this contradicts our choice of c, so $c=b$. This tells us that f is bounded on $[a,x]$, $x<b$. From the continuity of f, we know it is bounded on $[b-\varepsilon, b]$, $b-\varepsilon<b$. $\therefore$ f is bounded on $[a, b-\varepsilon]$ and $[b-\varepsilon, b]$ which means f is bounded on $[a,b]$.

QED

Extreme Value Theorem

By the lemma, f is bounded on $[a,b]$.

Set $M = \text{lub}\{f(x) : x \in [a,b]\}$

We have to show that there exists a c in $[a,b]$ such that $f(c)=M$

To do this, we set $g(x) = \dfrac{1}{M-f(x)}$

If $f(x) \neq M$, then $g(x)$ is cont on $[a,b]$ and bounded on $[a,b]$.

However, $g(x)$ cannot be bounded on $[a,b]$, which makes $f(x)=M$.

QED

Side Note *

The proof for m is similar

Intermediate Value Theorem:
If $f(x)$ is cont on $[a,b]$, then for any k strictly between $f(a)$ and $f(b)$, there exists at least one $c \in (a,b)$ such that $f(c)=k$

Proof:

Lemma:
If $f(x)$ is cont on $[a,b]$ and $f(a)<0<f(b)$, then $\exists\, c \in (a,b)$ for which $f(c)=0$

Proof

Let $f(a)<0<f(b)$

Since $f(a)<0$ and $f(x)$ is cont on $[a,b]$, we know a number E such that $f(x)$ is negative on $[a,E)$. Since $[a,E)$ is bounded above by b, b is an upper bound for $[a,E)$. By the LUB Axiom, this set has a Sup. Let's suppose $\text{Sup}\{E : f \text{ is negative on } [a,E)\} = c$.

We know that $c \leq b$, but since $f(x)$ is negative on $[a,E)$ and $f(b)>0$, $c \neq b$. Since $c<b$, $f(c)<0$ or $f(c)=0$. However, if $f(c)<0$, then there exists a number $c \pm \varepsilon$ for which $f(c\pm\varepsilon)<0$ and $c\pm\varepsilon$ becomes the new Sup. $\therefore f(c)=0$

QED

Proof of IVT:

Let $g(x)=f(x)-k$ for any $f(a)<k<f(b)$

Then, $g(a)=f(a)-k<0$ and $g(b)=f(b)-k>0$

From the lemma, we know $\exists\, x=c$, such that $g(c)=0$.

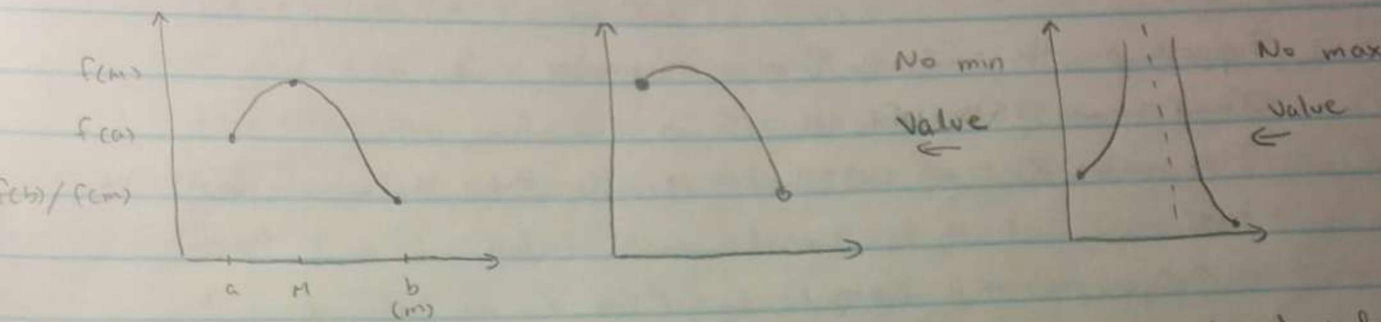
$\therefore g(c)=f(c)-k$

$0=f(c)-k$

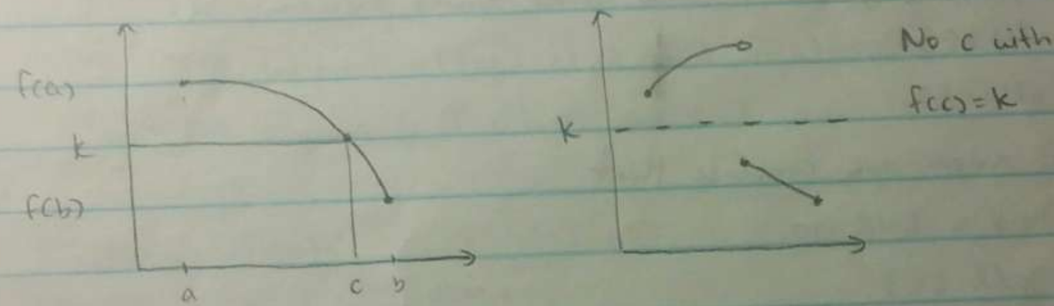
$f(c)=k$

QED

- Extreme Value Theorem: If f is continuous on a closed interval [a,b], then there exists values M and m in the interval [a,b] such that f(M) is the max value and f(m) is the min value.



- Intermediate Value Theorem: If f is cont. on a closed interval [a,b], then for any k between f(a) and f(b), there exists a $c \in (a,b)$ such that f(c) = k.



- A function can only change signs at a point x=c only if f(x)=0, undefined or discontinuous at x=c.

- Least Upper Bound Axiom/ Sup:
Every nonempty set of real numbers that has an upper bound has at a LUB. The LUB may not directly be in the set, but can be found from the pattern.
E.g. $S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \ldots\}$
The LUB of S is 1, because the numbers are approaching 1.
The largest element of a set, max S, exists if LUB S exists and LUB $S \in S$.

- If M is the LUB of set S and ε is a positive number, then there exists at least 1 number s in S such that $M-\varepsilon < s \leq M$.

Proof: Let $\varepsilon > 0$

$M-\varepsilon < S$

Suppose on the contrary there is no number in S.

That means $x \leq M-\varepsilon$ for all $x \in S$.

This makes $M-\varepsilon$ an LUB that is smaller than M, which can't be true.

$\therefore$ Our supposition is wrong

QED

- Every nonempty set of real numbers that has a lower bound has a GLB or Inf.

Proof: Suppose S in nonempty and has a lower bound k.

$k \leq s$, for all $s \in S$

$-s \leq -k$, for all $s \in S$

From the LUB axiom, we conclude that

$\{-s : s \in S\}$ has a LUB, m.

$-s \leq m$ for all $s \in S$

$-m \leq s$ for all $s \in S$

Suppose $-m < m_1 \leq s$, for all $s \in S$.

$-s \leq -m_1 < m$ for all $s \in S$

Thus m would not be LUB of $\{-s : s \in S\}$

- If m is the GLB of set S and ε is a positive number, then there exists a number s in S such that $m \leq s < m+\varepsilon$

- Bound Value Theorem:

If f is cont. on $[a,b]$, then f is bounded on $[a,b]$.

Proof: Consider $\{x : x \in [a,b]$ and f is bounded on $[a,x]\}$

Set $c = LUB\{x : f$ is bounded on $[a,x]\}$

Suppose $c<b$. From the continuity of f at c, f is bounded on $[c-\varepsilon, c+\varepsilon]$, $\varepsilon>0$. But this contradicts our choices of c. So $c=b$. This tells us f is bounded on $[a,x]$ with $x<b$. From the continunity of f, we know that f is bounded on some interval of the form $[b-\varepsilon, b]$. Since $b-\varepsilon<b$, f is bounded on $[a, b-\varepsilon]$ and $[b-\varepsilon, b]$. $\therefore$ f is bounded on $[a,b]$

- Properties of Limits

1. The sum/difference of 2 cont. functions is cont.
2. The product/quotient of 2 cont. functions is cont.
3. The composition $(f \circ g)$ of 2 cont functions is cont.
4. All polynomial functions are cont.

Limit Proofs

Let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

1. Constant Multiple Law

Prove $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$

$\lim_{x\to c} kf(x)$

$= \lim_{x\to c} k \cdot \lim_{x\to c} f(x)$

$= k \lim_{x\to c} f(x)$

2. Addition Law

Prove $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

$\lim_{x\to c} f(x) = L$ means $\forall \varepsilon_1 > 0, \exists d_1 > 0 \mid 0<|x-c|<d_1 \Rightarrow |f(x)-L|<\varepsilon_1$

Let ε_1 be $\frac{\varepsilon}{2}$

$\lim_{x\to c} g(x) = M$ means $\forall \varepsilon_2 > 0, \exists d_2 > 0 \mid 0<|x-c|<d_2 \Rightarrow |f(x)-M|<\varepsilon_2$

Let ε_2 be $\frac{\varepsilon}{2}$

Choose d to be $\min(d_1, d_2)$

Then, both $f(x)$ and $g(x)$ are within $\frac{\varepsilon}{2}$ of M and L.

$$\begin{array}{rl} & -\frac{\varepsilon}{2} < f(x)-L < \frac{\varepsilon}{2} \\ + & -\frac{\varepsilon}{2} < g(x)-M < \frac{\varepsilon}{2} \\ \hline & -\varepsilon < f(x)+g(x)-(L+M) < \frac{\varepsilon}{2} \end{array}$$

QED

3. Product Rule * Optional

Prove $\lim_{x\to c}[f(x)\cdot g(x)] = LM$

$\lim_{x\to c}[f(x)-L]=0$ and $\lim_{x\to c}[g(x)-L]=0$

Proof: $\lim_{x\to c}[f(x)-L]$

$= \lim_{x\to c} f(x) - \lim_{x\to c} L$

$= L-L$

$=0$

The same logic applies to $\lim_{x\to c}[g(x)-M]=0$.

Now, we have to prove $\lim_{x\to c} [f(x)-L][g(x)-M]=0$

$\lim_{x\to c} [f(x)-L]=0$ means:

$\forall \varepsilon_1 > 0, \exists d_1 > 0 \mid 0<|x-c|<d_1 \rightarrow |f(x)-L|<\varepsilon_1$

$\lim_{x\to c} [g(x)-M]=0$ means:

$\forall \varepsilon_2 > 0, \exists d_2 > 0 \mid 0<|x-c|<d_2 \rightarrow |g(x)-M|<\varepsilon_2$

Let $\varepsilon_1 = \sqrt{\varepsilon}$ and $\varepsilon_2 = \sqrt{\varepsilon}$

Choose d to be $\min(d_1, d_2)$

Then: $|f(x)-L|<\sqrt{\varepsilon}$ and $|g(x)-M|<\sqrt{\varepsilon}$

Multiplying the two together, we get $|(f(x)-L)(g(x)-M)|<\varepsilon$

$\therefore \lim_{x\to c} [f(x)-L][g(x)-M]=0$

If we multiply $[f(x)-L][g(x)-M]$, we get $f(x)g(x) - f(x)(M) - g(x)(L) + LM$

$f(x)g(x) = [f(x)-L][g(x)-M] + f(x)(M) + g(x)(L) - LM$

$$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} \left[[f(x)-L][g(x)-M] + f(x)M + g(x)L - LM \right]$$

$$= \lim_{x\to c} [f(x)-L][g(x)-M] + \lim_{x\to c} f(x)M + \lim_{x\to c} g(x)L - \lim_{x\to c} LM$$

$$= 0 + LM + LM - LM$$

$$= LM$$

QED

4. Difference Law

Prove $\lim_{x \to c} [f(x) - g(x)] = L - M$

$$\lim_{x \to c} [f(x) - g(x)]$$

$$= \lim_{x \to c} [f(x) + (-g(x))]$$

$$= \lim_{x \to c} f(x) + \lim_{x \to c} (-g(x))$$

$$= \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

$$= L - M$$

QED

5. Reciprocal Law

Prove $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}$, $M \neq 0$

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x - c| < d \to \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right|$$

$$= \left| \frac{M - g(x)}{M(g(x))} \right| \qquad (|M - g(x)| = |g(x) - M|) \text{ Side Note}$$

$$= \left| \frac{1}{M} \right| \left| \frac{1}{g(x)} \right| \left| g(x) - M \right|$$

Choose d_1 such that $|g(x) - M| < \frac{|M|}{2} \to |g(x)| > \frac{|M|}{2} \to \frac{1}{|g(x)|} < \frac{2}{|M|}$

Choose d_2 such that $|g(x) - M| < \frac{M^2 \varepsilon}{2}$

Choose d to be $\min(d_1, d_2)$

$$\left| \frac{1}{M} \right| \left| \frac{1}{g(x)} \right| \left| g(x) - M \right| < \frac{2}{M^2} \left(\frac{M^2}{2}\right)(\varepsilon)$$

$$< \varepsilon$$

QED

6. Quotient Law * Optional

Prove $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$

We can rewrite $\lim_{x \to c} \frac{f(x)}{g(x)}$ as $\lim_{x \to c} f(x)\left(\frac{1}{g(x)}\right)$

$\lim_{x \to c} f(x)\left(\frac{1}{g(x)}\right)$

$= \lim_{x \to c} f(x) \cdot \boxed{\lim_{x \to c} \frac{1}{g(x)}}$ ← We proved this in the reciprocal Law

$= L \cdot \frac{1}{M}$

$= \frac{L}{M}$ QED

7. Continunity of power function

Prove that $f(x) = x^k$ is continuous on $(-\infty, \infty)$

We have to prove $f(x)$ is cont on $(-\infty, \infty)$ first

1. Dom $f(x) = (-\infty, \infty)$
2. $f(c) = c^k$
3. $\lim_{x \to c} x^k = f(c) = c^k$

By the product rule, $\lim_{x \to c} x^k = \lim_{x \to c} x \cdot \lim_{x \to c} x \ldots$

$= c \cdot c \cdot c \ldots$

$= c^k$

QED

* Side Note *

We can also prove the continunity of $f(x) = x^{-k}$ and $f(x) = x^{p/q}$ using the same strategy.

9. Squeeze Theorem

Let $p>0$. Suppose that $\forall x$ such that $0<|x-c|<p$, $h(x) \le f(x) \le g(x)$.
If $\lim_{x\to c} h(x) = L$ and $\lim_{x\to c} g(x) = L$, then $\lim_{x\to c} f(x) = L$

Proof. $\forall \varepsilon>0, \exists d_1>0 \mid 0<|x-c|<d_1 \to |h(x)-L|<\varepsilon$

$\forall \varepsilon>0, \exists d_2>0 \mid 0<|x-c|<d_2 \to |g(x)-L|<\varepsilon$

Choose d to be $\min(d_1, d_2)$

Then, $|h(x)-L|<\varepsilon$ and $|g(x)-L|<\varepsilon$

$\downarrow$ $\downarrow$

$-\varepsilon < h(x)-L < \varepsilon$ $\quad$ $-\varepsilon < g(x)-L < \varepsilon$

$L-\varepsilon < h(x) < L+\varepsilon$ $\quad$ $L-\varepsilon < g(x) < L+\varepsilon$

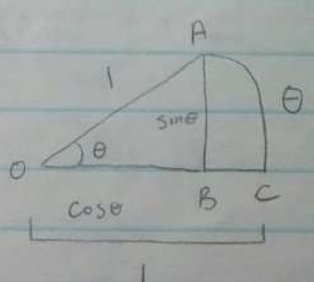
Since $L-\varepsilon \le h(x) \le f(x) \le g(x) \le L+\varepsilon$,

then $L-\varepsilon \le f(x) \le L+\varepsilon$

QED

10. Prove the continuity of $f(x) = \sin(x)$

1. Domain $\sin(x) = (-\infty, \infty)$
2. $f(c) = \sin(c)$
3. Find $\lim_{x\to c} \sin x = \sin c$



$\sin\theta = |AB|$

$0 \le |AB| \le \theta$

$\lim_{\theta\to 0} 0 = 0$ and $\lim_{\theta\to 0} \theta = 0$

$\therefore$ By S.T., $\lim_{\theta\to 0} \sin\theta = 0$

We also have to prove $\lim_{\theta\to 0} \cos\theta = 1$

$0 \le |BC| \le \theta$

$0 \le |1-\cos\theta| \le \theta$

$-1 \le -\cos\theta \le \theta-1$

$1 \ge \cos\theta \ge 1-\theta$

$\lim_{\theta\to 0} 1 = 1$ and $\lim_{\theta\to 0} 1-\theta = 1$

$\therefore$ By S.T., $\lim_{\theta\to 0} \cos\theta = 1$

Now, sub $x = c+h$ in. If $x \to c$, then $h \to 0$

$\lim_{x \to c} \sin x$

$= \lim_{h \to 0} \sin(c+h)$

$= \lim_{h \to 0} (\sin c \cosh + \sinh \cos c)$

$= \lim_{h \to 0} \sin c \cdot \cosh + \lim_{h \to 0} \sinh \cos c$

$= \lim_{h \to 0} \sin c \cdot \lim_{h \to 0} \cosh + \lim_{h \to 0} \sinh \cdot \lim_{h \to 0} \cos c$

$= \sin c$

QED

11. Prove the continuity of $f(x) = e^x$

We define e to be the number that $(1+h)^{\frac{1}{h}}$ approaches as h approaches 0.

$\lim_{h \to 0} (1+h)^{\frac{1}{h}} = e$ and $\lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) = 1$

1. Dom $e^x = (-\infty, \infty)$
2. $f(c) = e^c$
3. $\lim_{x \to c} e^x = e^c$

Sub $x = c+h$

$\lim_{h \to 0} e^x = \lim_{h \to 0} e^{c+h}$

$= \lim_{h \to 0} e^c \cdot \lim_{h \to 0} e^h$

$= e^c \cdot \lim_{h \to 0} (e^h + 1 - 1)$

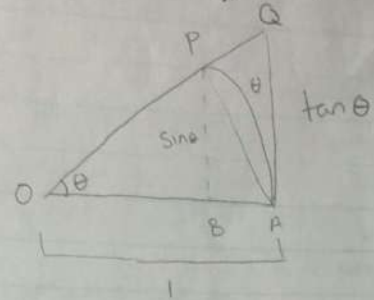
$= e^c \cdot \lim_{h \to 0} \left(\frac{e^h - 1}{h} h + 1\right)$

$= e^c \left(\lim_{h \to 0} \frac{e^h - 1}{h} \cdot \lim_{h \to 0} h + \lim_{h \to 0} 1\right)$

$= e^c$

QED

12 Prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$



Area $\triangle OPA = \frac{\sin\theta}{2}$

Area Sector OPA $= \frac{\theta}{2}$

Area of OPA $\leq$ Area of Sector OPA $\leq$ Area OQA

$$\frac{\sin\theta}{2} \leq \frac{\theta}{2} \leq \frac{\tan\theta}{2}$$

$$\sin\theta \leq \theta \leq \tan\theta$$

$$1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$

$$1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta$$

$\lim_{x \to 0} 1 = 1$ and $\lim_{x \to 0} \theta = 1$

$\therefore$ By S.T., $\lim_{x \to 0} \frac{\sin\theta}{\theta} = 1$

QED

13 Prove $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

$$\lim_{x\to 0} \frac{1-\cos x}{x}\left(\frac{1+\cos x}{1+\cos x}\right)$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x}$$

$= 0$

QED

14 Prove if $\lim_{x\to c} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{0^+}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$

$\lim_{x\to c} f(x) = 1$ means

$\forall \varepsilon_1 > 0, \exists d_1 > 0 \mid 0 < |x-c| < d_1 \to |f(x)-1| < \varepsilon_1$

$\Downarrow$

$1-\varepsilon_1 < f(x) < 1+\varepsilon_1$

$\lim_{x\to c} g(x) = 0$ means

$\forall \varepsilon_2 > 0, \exists d_2 > 0 \mid 0 < |x-c| < d_2 \to |g(x)| < \varepsilon_2$

$\Downarrow$

$-\varepsilon_2 < g(x) < \varepsilon_2$

Choose d to be $\min(d_1, d_2)$.

That means $\frac{f(x)}{g(x)} > \frac{1-\varepsilon_1}{\varepsilon_2}$

Let ε_1 be $\frac{1}{2}$

Let ε_2 be $\frac{1}{2M}$

$$\frac{f(x)}{g(x)} > \frac{1/2}{1/2M}$$

$$> M$$

QED

15. If $\lim_{x\to c} \frac{f(x)}{g(x)}$ is in the form of $\frac{1}{0^-}$, then $\frac{f(x)}{g(x)} = -\infty$

$\lim_{x\to c} f(x) = 1$ means

$\forall \varepsilon_1 > 0, \exists d_1 > 0 \mid 0 < |x-c| < d_1 \Rightarrow 1-\varepsilon_1 < f(x) < 1+\varepsilon_1$

$\lim_{x\to c} g(x) = 0$ means

$\forall \varepsilon_2 > 0, \exists d_2 > 0 \mid 0 < |x-c| < d_2 \Rightarrow -\varepsilon_2 < g(x) < \varepsilon_2$

Choose d to be $\min(d_1, d_2)$.

Then $\frac{f(x)}{g(x)} > \frac{1-\varepsilon_1}{\varepsilon_2}$

Let ε_1 be $\frac{1}{2}$.

Let ε_2 be $\frac{1}{2M}$

$\frac{f(x)}{g(x)} > M$

QED

16 If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{\infty}$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

$\lim_{x\to\infty} f(x) = 1$ means:

$\forall \varepsilon_1 > 0, \exists N_1 > 0 \mid x > N_1 \Rightarrow 1-\varepsilon_1 < f(x) < 1+\varepsilon_1$

$\lim_{x\to\infty} g(x) = \infty$ means:

$\forall M > 0, \exists N_2 > 0 \mid x > N_2 \Rightarrow g(x) > M$

Choose N to be $\max(N_1, N_2)$

That means $\left|\frac{f(x)}{g(x)}\right| < \frac{1+\varepsilon_1}{M}$

Let $\varepsilon_1 = 1$ and let $M = \frac{2}{\varepsilon}$

$\left|\frac{f(x)}{g(x)}\right| = \varepsilon$

QED

17 If $\lim_{x \to -\infty} \frac{f(x)}{g(x)}$ is in the form of $\frac{1}{-\infty}$, then $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = 0$

$\lim_{x \to -\infty} f(x) = 1$ means

$\forall \varepsilon_1 > 0, \exists N_1 < 0 \mid x < -N_1 \to 1 - \varepsilon_1 < f(x) < 1 + \varepsilon_1$

$\lim_{x \to -\infty} g(x) = -\infty$ means

$\forall M < 0, \exists N_2 < 0 \mid x < -N_2 \to g(x) < -M$

Choose N to be $\max(N_1, N_2)$

$$\left| \frac{f(x)}{g(x)} \right| < \frac{1 + \varepsilon_1}{M}$$

Let $\varepsilon_1 = 1$ and let $\varepsilon_2 = -2/\varepsilon$

$$\left| \frac{f(x)}{g(x)} \right| = M$$

QED

18. Continunity of Composition Functions

If g is cont at c and f is cont at $g(c)$, then $f \circ g$ is cont. at c.

WTS: $\forall \varepsilon > 0, \exists \delta > 0 \mid 0 < |x - c| < \delta \to |f(g(x)) - f(g(c))| < \varepsilon$

To show g is cont at c:

$\exists \delta > 0 \mid 0 < |x - c| < \delta \to |g(x) - g(c)| < \delta_1$ 1

To show f is cont at $g(c)$:

$\exists \delta_1 > 0 \mid 0 < |g(x) - g(c)| < \delta_1 \to |f(g(x)) - f(g(c))| < \varepsilon$ 2

Combining 1 and 2, we get

$\forall \varepsilon > 0, \exists \delta > 0 \mid 0 < |x - c| < \delta \to |f(g(x)) - f(g(c))| < \varepsilon$

QED

* Prove $\lim_{x\to\infty} (1+\frac{k}{x})^x = e^k$

$$e^{x\ln(1+\frac{k}{x})} = (1+\frac{k}{x})^x$$

$$\lim_{x\to\infty} (1+\frac{k}{x})^x = \lim_{x\to\infty} e^{x\ln(1+\frac{k}{x})}$$

$$= e^{\lim_{x\to\infty} x\ln(1+\frac{k}{x})}$$

$$= e^{\lim_{x\to\infty} \frac{\ln(1+\frac{k}{x})}{\frac{1}{x}}}$$

$$= e^{\lim_{x\to\infty} \frac{k}{1+\frac{k}{x}}}$$

$$= e^k$$

QED

* Prove $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

let $y = (1+x)^{\frac{1}{x}}$

$$\ln(y) = \ln((1+x)^{\frac{1}{x}})$$
$$= \frac{1}{x}\ln(1+x)$$
$$= \lim_{x\to 0} (\tfrac{1}{x})(\ln(1+x))$$

Using l'hôpital's Rule, we get

$$= \lim_{x\to 0} \frac{1}{1+x}$$

$$= 1$$

$$\ln(y) = 1$$
$$y = e$$

$\therefore \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

QED

Bisectional Method / Method of Bisections

- Used to find approx value of roots
- Recall that according to the IVT, a function can only change signs at roots or discontinuities.
- That means, if you know a function is cont with closed interval $[a,b]$, and you know that the function changes signs, there must be a root.

Eg.

$f(x) = x^3 + x^2 + 3x + 5$

Show that $f(x)$ has a root in $[-2,0]$.

$f(0) = 5$

$f(-2) = -5$

Since $f(x)$ is a polynomial, it is cont in $[-2,0]$.

Since $f(x)$ changes sign between 0 and -2, we know there must be a root between 0 and -2.

To find the root, we'll find the midpoint of 0 and -2 and see if its positive or negative.

$f(-1) = 2$

This means the root is between -1 and -2

$f(-1.5) = -0.625$

This means the root is between -1 and -1.5

Since the numbers are getting harder to calculate, I will stop here.

P.S. The root is -1.4, which concludes that my work is right.

Prove $\lim_{x \to c} k = k$

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \rightarrow |k-k| < \varepsilon$

Since $k-k=0$ and $\varepsilon > 0$, this statement is always true. QED

Prove $\lim_{x \to c} x = c$

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \rightarrow |x-c| < \varepsilon$

Choose $d = \varepsilon$

$|x-c| < \varepsilon$

But $|x-c| < d$

$d = \varepsilon$

$\varepsilon = \varepsilon$

QED

Prove $\lim_{x \to c} mx+b = mc+b$

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \rightarrow |mx+b-mc-b| < \varepsilon$

$|mx+b-mc-b|$

$= |mx-mc|$

$= m|x-c|$

$= md$

$md = \varepsilon$

$d = \varepsilon/m$

$|mx+b-mc-b| < \varepsilon$

$|mx-mc| < \varepsilon$

$m|x-c| < \varepsilon$

$m(\varepsilon/m)$

$= \varepsilon$

QED

Summary of Limit Laws

1. $\lim_{x \to c} k f(x) = k \lim_{x \to c} f(x)$

2. $\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

3. $\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, $\lim_{x \to c} g(x) \neq 0$

5. $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}$, $M \neq 0$

6. $\lim_{x \to c} [f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$, n is a positive int

7. $\lim_{x \to c} \sqrt[n]{f(x)} = \left[\sqrt[n]{\lim_{x \to c} f(x)}\right]$, n is a positive int

8. $\lim_{x \to c} k = k$

9. $\lim_{x \to c} x = c$

10. $\lim_{x \to c} (mx+b) = \lim_{x \to c} (mc+b)$

11. $\lim_{x \to c} Ax^k = Ac^k$

12. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

13. $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

14 $\lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x = e^k$

15 $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

Indeterminate Forms of Limits:

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

Non-Indeterminate Forms of Limits:

a) A limit in any of these forms is equal to 0

$\frac{1}{\infty}$, $\frac{0}{\infty}$, $\frac{0}{1}$, 0^∞, 0^1

b) A limit in any of these forms is equal to infinity.

$\frac{1}{0^+}$, $\frac{\infty}{0^+}$, $\frac{\infty}{1}$, $\infty + \infty$, $\infty \cdot \infty$, ∞^∞, ∞^1

Tips on evaluating limits:

1. Most trig limits can be solved using S.T.

E.g.

Evaluate $\lim_{x\to\infty} x\sin(\frac{1}{x})$

We know that $\sin(x) \in (-1, 1)$

$\therefore \sin(\frac{1}{x}) \in -1, 1$

$-1 \leq \sin(\frac{1}{x}) \leq 1$

$-x \leq x\sin(\frac{1}{x}) \leq x$

$\lim_{x\to\infty} -x = 0$ $\lim_{x\to\infty} x = 0$

$\therefore$ By S.T., $\lim_{x\to\infty} x\sin(\frac{1}{x}) = 0$

Lemma proof for IVT

1. If $f(x)$ is cont on closed interval $[a,b]$ and $f(a)<0<f(b)$ then there is a number c, such that if $c\in(a,b)$ then $f(c)=0$.

Proof:

Let $f(a)<0<f(b)$

Since $f(a)<0$ and $f(x)$ is cont on $[a,b]$, then there exists a number, m, such that $f(x)$ is negative on $[a,m)$. Set $[a,m)$ is bounded above by b, so b is an upper bound for $[a,m)$. By LUB Axiom, the set has a Supremum.

Suppose $\sup\{m: f \text{ is negative on } [a,m)\}=c$.

This means $c<b$ because $f(b)>0$ and $f(x)<0$ on $[a,m)$.

$\therefore c<0$ or $c=0$

$c<0$ can't be true because if it is true, then there exists a number, $c\pm\varepsilon$, for which $f(c\pm\varepsilon)<0$ and then $\sup\{m: f \text{ is negative on } [a,m)\}=c\pm\varepsilon$, which contradicts the supposition. $\therefore f(c)=0$.

QED

2. Prove IVT

Proof:

Let $g(x)=f(x)-k$, for any $f(a)<k<f(b)$

Then $g(a)=f(a)-k<0$ and $g(b)=f(b)-k>0$

From the proof above we know there exists an $x=c$ such that $g(c)=0$.

$$\therefore g(c)=f(c)-k$$
$$=0$$

$\therefore f(c)=k$

QED

Proving Derivative Rules

1. Constant Multiple Rule

Prove $(rf)'(x) = rf'(x)$

$$\lim_{h\to 0} \frac{r(f(x+h)) - r(f(x))}{h}$$

$$= \lim_{h\to 0} \frac{r(f(x+h) - f(x))}{h}$$

$$= r \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= rf'(x)$$

2. Sum Rule

Prove $(f+g)'(x) = f'(x) + g'(x)$

$$\lim_{h\to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

3. Difference Rule

Prove $(f-g)'(x) = f'(x) - g(x)$

$$\lim_{h\to 0} \frac{f(x+h) - g(x+h) - (f(x) - g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) - g(x+h) + g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) - g'(x)$$

4. Product Rule

Prove $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x) - f(x)g(x+h) + f(x)g(x+h)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)]g(x+h) + [g(x+h) - g(x)]f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} g(x+h) + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} f(x)$$

$$= f'(x)g(x) + f(x)g'(x)$$

5. Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = (f(x) \cdot [g(x)]^{-1})'$$

$$= \frac{f'(x)}{g(x)} + \frac{f(x)(-1)(g'(x))}{(g(x))^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

6. Show that $(e^x)' = e^x$

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^h e^x - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h}$$

$$= \lim_{h \to 0} e^x + \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x$$

7. Prove that if $f(x) = a^x$, then $f'(a) = a^x \ln a$

$a^x = e^{x \ln a}$

$(a^x)' = (e^{x \ln a})'$

$= e^{x \ln a} (x \ln a)'$

$= a^x \ln a$

8. Prove that if $f(x) = \log_a x$, for any $a > 0$, $a \neq 1$, then $f'(x) = \frac{1}{x \ln a}$

$x = a^{\log_a x}$

$x' = (a^{\log_a x})'$

$= a^{\log_a x} (\ln a)(\log_a x)'$

$1 = x \ln a (\log_a x)'$

$(\log_a x)' = \frac{1}{x \ln a}$

9. Prove that if $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$

$e^{f(x)} = e^{\ln x}$

$(e^{f(x)})' = (e^{\ln x})'$

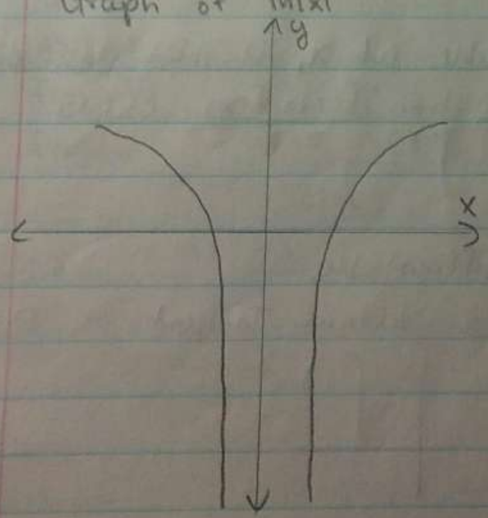
$e^{f(x)} \cdot f'(x) = x'$

$= 1$

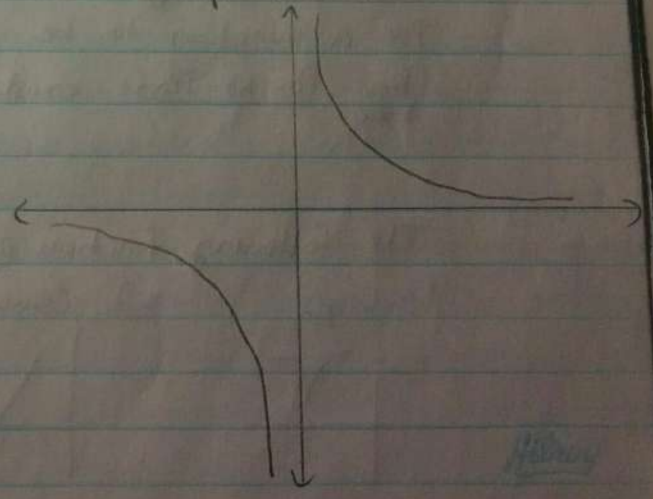
$f'(x) = \frac{1}{x}$

10 Prove if $f(x) = \ln|x|$, then $f'(x) = \frac{1}{x}$

Graph of $\ln|x|$



Graph of $\frac{1}{x}$



11. Chain Rule Proof

Prove $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \left(\frac{g(x+h) - g(x)}{g(x+h) - g(x)} \right)$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(g(x)) \cdot g'(x)$$

Equation of a tangent line

$f(x) = f(a) + f'(a)(x-a)$

Eg. Find the eqn of the tangent line to x^2+2x+1 at $x=2$.

$a = 2$

$f(a) = (2)^2 + 2(2) + 1$

$= 4+4+1$

$= 9$

$f'(a) = 2x+2$

$= 2(2)+2$

$= 6$

$f(x) = 9 + 6(x-2)$

$= 6x - 12 + 9$

$= 6x - 3$

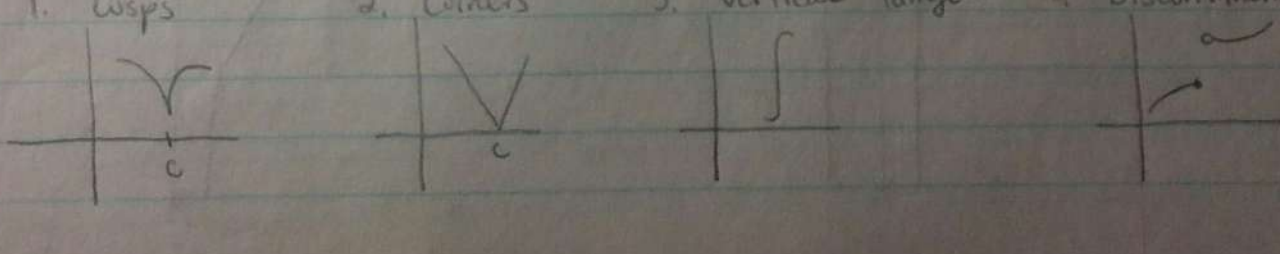
Diffrentiability:

For a function to be diffrentiable at x, it must pass either rule:

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists OR $\lim_{z \to x} \frac{f(z) - f(x)}{z - x}$ exists

The following functions are not diffrentiable

1. Cusps 2. Corners 3. Vertical Tangent 4. Discontinuity



12. Proof of Power Rule

Prove $f'(x) = n(x^{n-1})$

$(a+b)^n = a^n + na^{n-1}b \dots + b^n$

$$(x^n)' = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{x^n + nx^{n-1}h \dots + h^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h \dots + h^n}{h}$$

$$= nx^{n-1}$$

13. Proof of Constant Function

Show $f'(x) = 0$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= 0$$

Linearization and Newton's Method

1. Linearization
   1. $f(x) = f(a) + f'(a)(x-a)$
   2. $f(a+\Delta x) = f(a) + f'(a)\Delta x$

E.g. Find the linear approximation of $\sqrt{16.1}$

$a = 16$

$\Delta x = 0.1$

$f(x) = \sqrt{x}$

$$f(a+\Delta x) = \sqrt{16} + \frac{1}{2\sqrt{16}}(0.1)$$

$$= 4 + \frac{0.1}{8}$$

$$= 4.0125$$

$\Delta y = f(x+\Delta x) - f(x)$
$dy = f'(a)\,dx$

E.g. Find Δy and dy for the function $f(x) = \frac{1}{x+2}$ when $x=1$, $\Delta x = 0.01$

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= \frac{1}{1.01+2} - \frac{1}{1+2}$$
$$= \frac{1}{3.01} - \frac{1}{3}$$
$$= -0.0011074$$

$$dy = f'(a)\,dx$$
$$= \frac{-1}{(x+2)^2}(0.01) \leftarrow \Delta x = dx$$
$$= \frac{-0.01}{(1+2)^2}$$
$$= \frac{-0.01}{9}$$
$$\approx -0.0011$$

2. Newton's Method

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

E.g. Find at least 1 root of $x^3 - 2x - 5 = 0$ with the accuracy up to 3 decimal places.

$f(0) = -5$
$f(3) = 3^3 - 2(3) - 5$
$= 27 - 6 - 5$
$= 16$

$\therefore$ The root is between $x=0$ and $x=3$

Let $x=2$ be x_0

$x_1 = 2 - \left(\frac{-1}{10}\right)$
$= 2.1$

$x_2 = 2.1 - \left(\frac{0.061}{11.23}\right)$
$= 2.095$

$\therefore$ The root is approx 2.095

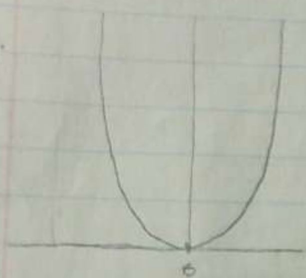
Eqn of secant line: $y - f(a) = \frac{f(a+h) - f(a)}{h}(x-a)$

Derivative = Slope of tangent = Instanteous Rate of Change

Relationships Btwn $f(x)$ and $f'(x)$:

1. For each x, the slope of $f(x)$ is the height of $f'(x)$
2. Where $f(x)$ has a horizontal tangent, $f'(x)$ has a root
3. Whenever $f(x)$ is increasing, $f'(x) > 0$
4. Whenever $f(x)$ is decreasing, $f'(x) < 0$
5. Whenever $f(x)$ has a steep slope, $f'(x)$ has a large magnitude
6. Whenever $f(x)$ has a shallow slope, $f'(x)$ has a small magnitude

E.g.

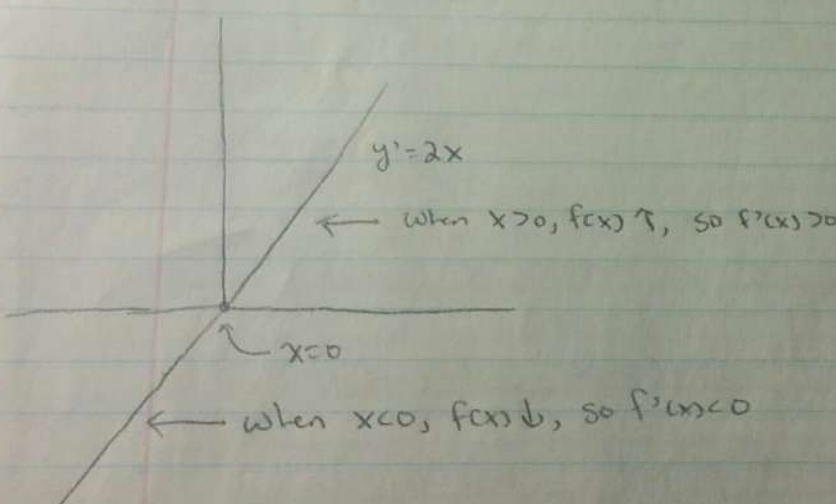


$y = x^2$

At $x = 0$ there is a horizontal tangent

If $x < 0$, y is decreasing

If $x > 0$, y is increasing



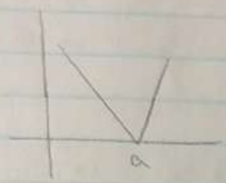
Left Derivative

$$f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

Right Derivative

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

When doing questions with piecewise functions or



, you have to check continuity and if left derivative = right derivative.

Differentiability implies continuity but continuity does NOT imply differentiability.

Week 9 Notes

Proof of Trig Derivatives

1. $(\sin x)' = \cos x$

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h\to 0} \frac{\sinh \cos x}{h}$$

$$= \lim_{h\to 0} \underbrace{\frac{\sin x(\cos h - 1)}{h}}_{\to \sin x(0)} + \lim_{h\to 0} \underbrace{\cos x\left(\frac{\sin h}{h}\right)}_{\to \cos x(1)}$$

$$= \cos x$$

2. $(\cos x)' = -\sin x$

$$(\cos x)' = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h}$$

$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \sin x$$

$$= -\sin x$$

3. $(\tan x)' = \sec^2 x$

$$(\tan x)' = \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \sin x(\cos(x+h))}{h(\cos x)(\cos(x+h))}$$

$$= \lim_{h\to 0} \frac{\cos x(\sin x \cos h + \cos x \sin h) - \sin x(\cos x \cos h - \sin x \sin h)}{h(\cos x)(\cos(x+h))}$$

$$= \lim_{h\to 0} \frac{\cos^2 x \sin h + \sin^2 x \sin h}{h(\cos x)(\cos(x+h))}$$

$$= \lim_{h\to 0} \frac{\sin h\,(\cos^2 x + \sin^2 x)}{\cos x\,(\cos(x+h))\,h}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

4. $(\sec x)' = \sec x \tan x$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)'$$

$$= \frac{-(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \sec x \ \tan x$$

5. $(\csc x)' = -\csc x \cdot \cot x$

$$(\csc x)' = \left(\frac{1}{\sin x}\right)'$$

$$= \frac{-\cos x}{\sin^2 x}$$

$$= -\csc x \cdot \cot x$$

6. $(\cot x)' = -\csc^2(x)$

$$(\cot x)' = \left(\frac{1}{\tan x}\right)'$$

$$= \frac{-\sec^2 x}{\tan^2 x}$$

$$= \frac{-1}{\cos^2 x}$$

$$= -\csc^2 x$$

Proof of Inverse Trig Functions

1. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$y = \arcsin(x)$

$x = \sin y$

$x' = (\sin y)'$

$1 = \cos y \cdot y'$

$y' = \frac{1}{\cos y}$

$= \frac{1}{\sqrt{1-\sin^2 y}} \leftarrow \cos x = \sqrt{1-\sin^2 x}$

$= \frac{1}{\sqrt{1-x^2}}$

2. $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$

$y = \arccos x$

$x = \cos y$

$x' = (\cos y)'$

$1 = -\sin(y) \cdot y'$

$y' = \frac{-1}{\sin y}$

$= \frac{-1}{\sqrt{1-x^2}}$

3. $(\arctan x)' = \frac{1}{1+x^2}$

$y = \arctan x$

$x = \tan y$

$x' = (\tan y)'$

$1 = \sec^2 y \cdot y'$

$y' = \frac{1}{\sec^2 y}$

$= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

4. $(\text{arc}\cot x)' = \frac{-1}{1+x^2}$

$y = \text{arc}\cot x$

$x = \cot y$

$x' = (\cot y)'$

$1 = -\csc^2 y \cdot y'$

$y' = \frac{-1}{\csc^2 y}$

$= \frac{-1}{1+\cot^2 y}$

$= \frac{-1}{1+x^2}$

5. $(\text{arc}\sec x)' = \frac{1}{|x|\sqrt{x^2-1}}$

$y = \text{arc}\sec x$

$x = \sec y$

$x' = (\sec y)'$

$1 = \sec y \cdot \tan y \cdot y'$

$y' = \frac{1}{\sec y \cdot \tan y}$

$= \frac{1}{|x|\sqrt{x^2-1}}$

6. $(\text{arc}\csc x)' = \frac{-1}{|x|\sqrt{x^2-1}}$

$y = \text{arc}\csc x$

$x = \csc y$

$x' = (\csc y)'$

$1 = -\csc y \cdot \cot y \cdot y'$

$y' = \frac{-1}{\csc y \cdot \cot y}$

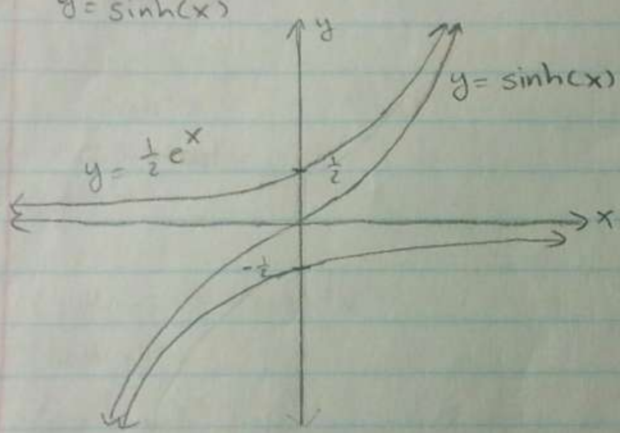
$= \frac{-1}{|x|\sqrt{x^2-1}}$

Hyperbolic Identities and Proofs

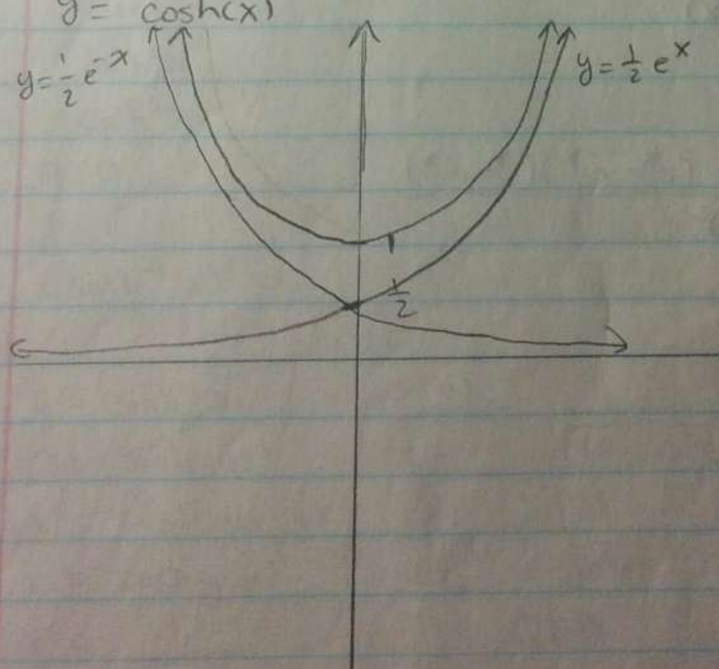
1. $\sinh(t) = \dfrac{e^{t} - e^{-t}}{2}$
2. $\cosh(t) = \dfrac{e^{t} + e^{-t}}{2}$
3. $\tanh(t) = \dfrac{\sinh(t)}{\cosh(t)}$
   $= \dfrac{e^{t} - e^{-t}}{e^{t} + e^{-t}}$
4. $\cosh^2 t - \sinh^2 t = 1$

Graph of hyperbolic functions

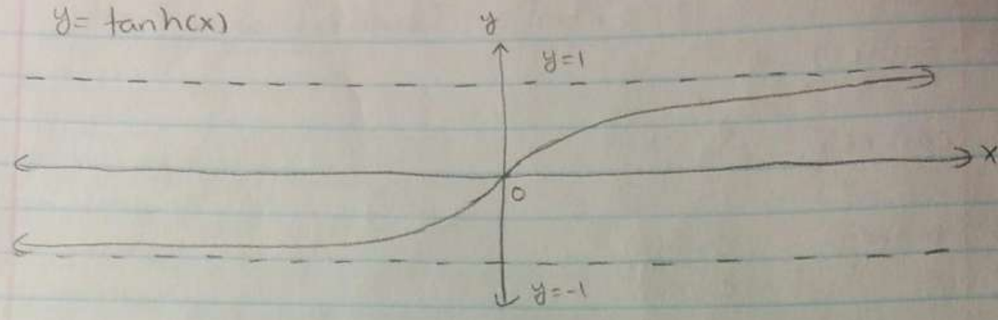
$y = \sinh(x)$



$y = \cosh(x)$



$y = \tanh(x)$



Prove

1. $(\sinh x)' = \cosh x$

$\left[\left(\frac{1}{2}\right)(e^{x} - e^{-x})\right]'$

$= \frac{1}{2}(e^{x} + e^{-x})$

$= \cosh(x)$

QED

2. $(\cosh x)' = \sinh x$

$\left[\frac{1}{2}(e^{x} + e^{-x})\right]'$

$= \frac{1}{2}(e^{x} - e^{-x})$

$= \sinh(x)$

QED

3. $(\tanh x)' = (\operatorname{sech}^2 x)$

$\left(\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\right)'$

$= \frac{(e^{x} + e^{-x})(e^{x} + e^{-x}) - (e^{x} - e^{-x})(e^{x} - e^{-x})}{(e^{x} + e^{-x})^2}$

$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$

$= \frac{1}{\cosh^2 x}$

$= \operatorname{sech}^2 x$

④

4. $(\text{sech } x)' = -\text{sech}x \cdot \tanh x$

$(\text{sech } x)'$

$= \left(\frac{1}{\cosh x}\right)'$

$= \left(\frac{-\sin x}{\cosh^2 x}\right)$

$= -\text{sech}x \cdot \tanh x$

QED

5. $(\text{csch } x)' = -\text{csch } x \cdot \coth x$

$(\text{csch } x)'$

$= \left(\frac{1}{\sinh x}\right)'$

$= \frac{-\cosh x}{\sinh^2 x}$

$= -\text{csch} x \cdot \coth x$

QED

6. $(\coth x)' = -\text{csch}^2 x$

$(\coth x)'$

$= \left(\frac{1}{\tanh x}\right)'$

$= \frac{-\text{sech}^2 x}{\tanh^2 x}$

$= -\text{csch}^2 x$

QED

7. $(\sinh^{-1} x)' = \frac{1}{\sqrt{x^2+1}}$

$y = \sinh^{-1} x$

$x = \sinh(y)$

$x' = \cosh(y) \cdot y'$

$1 = \cosh y \cdot y'$

$y' = \frac{1}{\cosh y}$

$\rightarrow y' = \frac{1}{\sqrt{1+\sinh^2(\sinh^{-1} x)}}$

$= \frac{1}{\sqrt{x^2+1}}$

QED

8. $(\cosh^{-1} x)' = \dfrac{1}{\sqrt{x^2-1}}$

$y = \cosh^{-1} x$

$x = \cosh y$

$x' = (\cosh y)'$

$1 = \sinh y \cdot y'$

$y' = \dfrac{1}{\sinh y}$

$= \dfrac{1}{\sqrt{\sinh^2(\cosh^{-1} x) - 1}}$

$= \dfrac{1}{\sqrt{x^2-1}}$

9. $(\tanh^{-1} x)' = \dfrac{1}{1-x^2}$

$y = \tanh^{-1} x$

$x = \tanh y$

$x' = (\tanh y)'$

$1 = \operatorname{sech}^2 y \cdot y'$

$y' = \dfrac{1}{\operatorname{sech}^2 y}$

$= \dfrac{1}{\operatorname{sech}^2(\tanh^{-1} x)}$

$= \dfrac{1}{1-x^2}$

10. $(\operatorname{sech}^{-1} x)' = \dfrac{-1}{x\sqrt{1-x^2}}$

$y = \operatorname{sech}^{-1} x$

$x = \operatorname{sech} y$

$x' = (\operatorname{sech} y)'$

$1 = -\operatorname{sech} y \cdot \tanh y \cdot y'$

$y' = \dfrac{-1}{\operatorname{sech} y \cdot \tanh y}$

$= \dfrac{-1}{\operatorname{sech}(\operatorname{sech}^{-1} x) \cdot \tanh(\operatorname{sech}^{-1} x)}$

$\to y' = \dfrac{-1}{x\sqrt{1-x^2}}$

11 $(\operatorname{csch}^{-1} x)' = \dfrac{-1}{|x|\sqrt{1+x^2}}$

$y = \operatorname{csch}^{-1} x$

$x = \operatorname{csch} y$

$x' = (\operatorname{csch} y)'$

$1 = -\operatorname{csch} y \cdot \coth y \cdot y'$

$y' = \dfrac{-1}{\operatorname{csch} y \cdot \coth y}$

$= \dfrac{-1}{\operatorname{csch}(\operatorname{csch}^{-1} x) \cdot \coth(\operatorname{csch}^{-1} x)}$

$= \dfrac{-1}{|x|\sqrt{1+x^2}}$

12. $(\coth^{-1} x)' = \dfrac{1}{1-x^2}$

$y = \coth^{-1} x$

$x = \coth y$

$x' = (\coth y)'$

$1 = -\operatorname{csch}^2 y \cdot y'$

$y' = \dfrac{-1}{\operatorname{csch}^2 y}$

$= \dfrac{-1}{\operatorname{csch}^2(\coth^{-1} x)}$

$= \dfrac{-1}{x^2-1} = \dfrac{1}{1-x^2}$

13. $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ for any x

let $y = \sinh^{-1} x$

$x = \sinh y$

$= \dfrac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$= e^{\sinh^{-1} x} - e^{-\sinh^{-1} x}$

$2xe^{\sinh^{-1} x} = (e^{\sinh^{-1} x} - e^{-\sinh^{-1} x})e^{\sinh^{-1} x}$

$= (e^{\sinh^{-1} x})^2 - 1$

Let u be $e^{\sinh^{-1}x}$

$0 = u^2 - 2xu - 1$

$u = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{2x \pm \sqrt{4x^2 + 4}}{2}$

$= x \pm \sqrt{x^2+1}$

However, since $x - \sqrt{x^2+1} < 0$, we reject this case

$\therefore \quad u = x + \sqrt{x^2+1}$

$e^{\sinh^{-1}x} = x + \sqrt{x^2+1}$

$\ln(u) = \ln(x + \sqrt{x^2+1})$

$\sinh^{-1}x = \ln(x + \sqrt{x^2+1})$

14. $\cosh^{-1}x = \ln(x + \sqrt{x^2-1})$ for $x \geq 1$ (Proof is similar to 13)

Let $y = \cosh^{-1}x$

$x = \cosh y$

$= \dfrac{e^{y} + e^{-y}}{2}$

$2x = e^{\cosh^{-1}x} + e^{-\cosh^{-1}x}$

$2xe^{\cosh^{-1}x} = (e^{\cosh^{-1}x} + e^{-\cosh^{-1}x})e^{\cosh^{-1}x}$

$= (e^{\cosh^{-1}x})^2 + 1$

Let $u = e^{\cosh^{-1}x}$

$0 = u^2 - 2xu + 1$

$u = \dfrac{2x \pm \sqrt{4x^2 - 4}}{2}$

$= x \pm \sqrt{x^2-1}$

Since $x - \sqrt{x^2-1} < 0$, we reject this case

$u = x + \sqrt{x^2-1}$

$\ln(u) = \ln(x + \sqrt{x^2-1})$

$\cosh^{-1}x = \ln(x + \sqrt{x^2-1})$

15 $\tanh^{-1} x = \frac{1}{2}\left(\ln\left(\frac{1+x}{1-x}\right)\right)$ for $-1<x<1$

Let $y = \tanh^{-1} x$

$$x = \tanh y$$
$$= \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$
$$x(e^{y}+e^{-y}) = e^{y} - e^{-y}$$
$$xe^{y} + xe^{-y} = e^{y} - e^{-y}$$
$$e^{y}(xe^{y} + xe^{-y}) = e^{y}(e^{y} - e^{-y})$$
$$xe^{2y} + x = e^{2y} - 1$$
$$0 = e^{2y} - xe^{2y} - x - 1$$
$$= e^{2y}(1-x) - x - 1$$
$$x+1 = e^{2y}(1-x)$$
$$e^{2y} = \frac{x+1}{1-x}$$
$$\ln(e^{2y}) = \ln\left(\frac{x+1}{1-x}\right)$$
$$2y = \ln\left(\frac{1+x}{1-x}\right)$$
$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Side Notes

- when doing proofs for inverse trig or inverse hyperbolic functions, start out with $y =$ (inverse trig) x
  $x =$ trig (y).
- trig func (trig inverse func x) $= x$
  Eg. $\sin(\sin^{-1} x) = x$
  <u>BUT</u>! trig inverse func (trig func) $\neq x$
  $\sin^{-1}(\sin x) \neq x$

- Principle Functions:
  $\sin(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
  $\cos(x) \in [0, \pi]$
  $\tan(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$
  $\csc(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $x \neq 0$
  $\sec(x) \in (0, \pi)$ $x \neq \frac{\pi}{2}$
  $\cot(x) \in (0, \pi)$

Summary of Derivatives

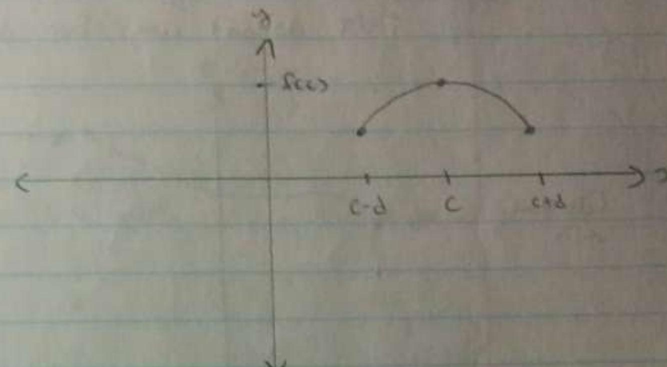
1. $\frac{d}{dx} c = 0$, c is a constant
2. $\frac{d}{dx} x = 1$
3. $\frac{d}{dx} cx = c$, c is a constant
4. $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) \cdot f(x)g'(x)$
6. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
7. $\left[(f(x))^n\right]' = n(f(x))^{n-1} \cdot f'(x)$
8. $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
9. If $f(x) = a^x$, for any $a > 0$, $a \neq 1$, then $f'(x) = a^x \cdot \ln a \cdot x'$
10. $(e^x)' = e^x \cdot x'$
11. If $f(x) = \log_b x$, for any $b > 0$, $b \neq 1$ then $f'(x) = \frac{1}{x \ln a} \cdot x'$
12. $f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x} \cdot x'$
13. $f(x) = \ln|x| \rightarrow f'(x) = \frac{x'}{x}$ ✱ Special Case

14. $(\sin x)' = \cos x \cdot x'$ $\qquad (\sinh(x))' = \cosh(x) \cdot x'$
15. $(\cos x)' = -\sin x \cdot x'$ $\qquad (\cosh(x))' = \sinh(x) \cdot x'$
16. $(\tan x)' = \sec^2 x \cdot x'$ $\qquad (\tanh x)' = \operatorname{sech}^2 x \cdot x'$
17. $(\sec x)' = \sec x \cdot \tan x \cdot x'$ $\qquad (\operatorname{sech} x)' = -\operatorname{sech} x \cdot \tanh x \cdot x'$
18. $(\csc x)' = -\csc x \cdot \cot x \cdot x'$ $\qquad (\operatorname{csch} x)' = -\operatorname{csch} x \cdot \coth x \cdot x'$
19. $(\cot x)' = -\csc^2 x \cdot x'$ $\qquad (\coth x)' = -\operatorname{csch}^2 x \cdot x'$
20. $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \cdot x'$ $\qquad (\sinh^{-1} x)' = \frac{1}{\sqrt{1+x^2}} \cdot x'$
21. $(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}} \cdot x'$ $\qquad (\cosh^{-1} x)' = \frac{1}{\sqrt{x^2-1}} \cdot x'$
22. $(\tan^{-1} x)' = \frac{1}{1+x^2} \cdot x'$ $\qquad (\tanh^{-1} x)' = \frac{1}{1-x^2} \cdot x'$
23. $(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}} \cdot x'$ $\qquad (\operatorname{sech}^{-1} x)' = \frac{-1}{x\sqrt{1-x^2}} \cdot x'$
24. $(\csc^{-1} x)' = \frac{-1}{|x|\sqrt{x^2-1}} \cdot x'$ $\qquad (\operatorname{csch}^{-1} x)' = \frac{-1}{|x|\sqrt{1+x^2}} \cdot x'$
25. $(\cot^{-1} x) = \frac{-1}{1+x^2} \cdot x'$ $\qquad (\coth^{-1} x)' = \frac{1}{1-x^2} \cdot x'$

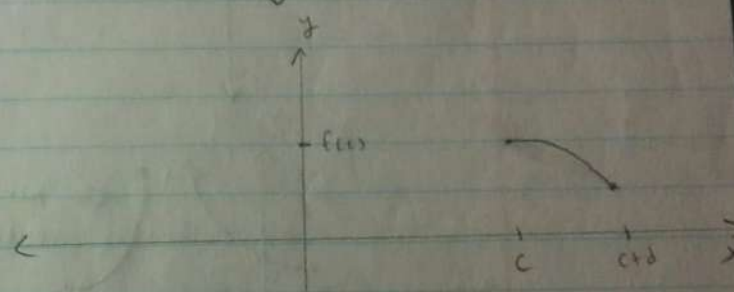
Week 10/11 Notes

1. Fermat's Theorem for Local Extrema:
If $f(x)$ has a local max at an interior point c and $f'(c)$ exists, then $f'(c) = 0$.
Proof:
From the definiton $f(x)$ has a local max at c if $f(x) \leq f(c)$ on $x \in (c-d, c+d)$.

$$f'_+(c) = \begin{cases} x \in (c, c+d) \\ f(x) - f(c) \leq 0 \\ x - c \geq 0 \end{cases}$$



$$f'_+(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leq 0$$

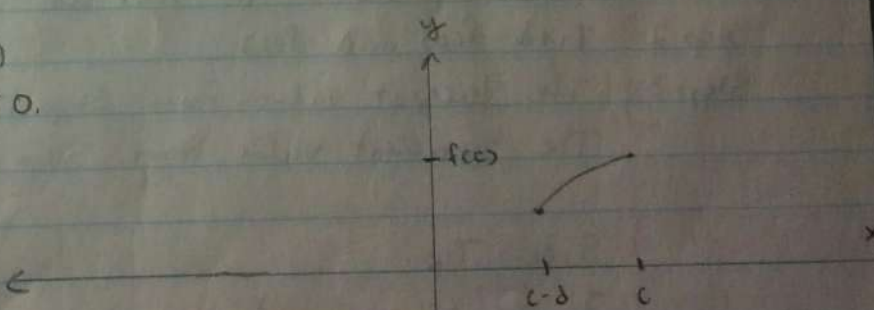


$$f'_-(c) = \begin{cases} x \in (c-d, c) \\ f(x) - f(c) \geq 0 \\ x - c \leq 0 \end{cases}$$

Since $x \in (c, c+d)$ then $x > c$.
However, since $f(c)$ is a local max, $f(x) \leq f(c)$

$$f'_-(c) = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \geq 0$$

However, because $f'(c)$ exists, $f'_+(c) = f'_-(c) = 0$.
QED



* Note *
The inverse is false.
$f'(c) = 0$ doesn't mean $f(c)$ is a local max.
$f(c)$ can be a local min.

Since $x \in (c-d, c)$, $x \leq c$.
Since $f(c)$ is a local ma[x]
$f(x) \leq f(c)$

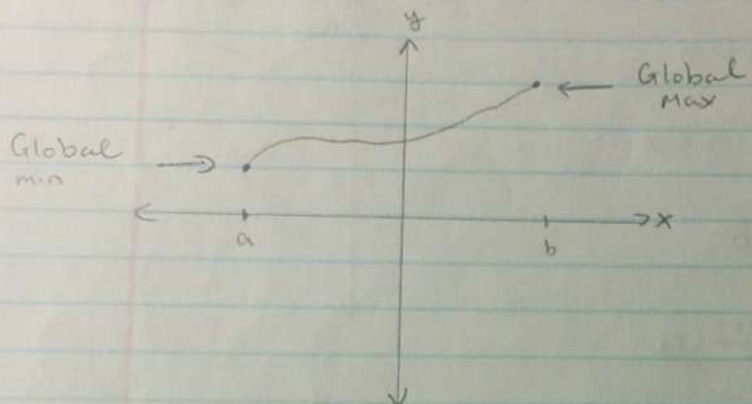
2 Extreme Value Theorem (EVT):

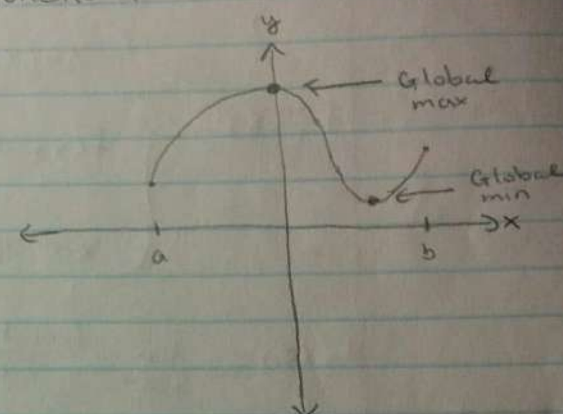
If f(x) is cont on closed interval [a,b], then f(x) attains an absolute max or global max and an absolute min or global min at some numbers in [a,b]

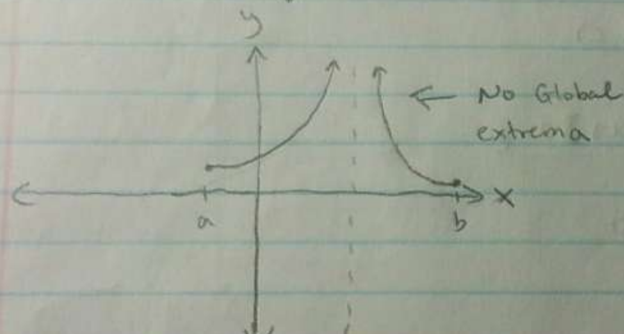
abs max = global max

abs min = global min

EVT doesn't work for discont functions.







Use the Closed Interval Method to find global extrema

Step 1: Find the values of f(x) at the critical

Step 2: Find f(a) and f(b)

Step 3: The largest value from Step 1 and 2 is the abs max

The smallest value from Step 1 and 2 is the abs min

Side Tip:

If one or more of the endpoints is open, eg. (a,b] or [a,b) or (a,b) instead of [a,b], do step 1 the same way, but for step 2, you find the limit for all open intervals. If an open interval is greater or less than all critical points, an abs max or min won't exist.

E.g. Find the abs max and abs min of the function
$f(x) = x^3 - 27x + 1$ at $[-1, 6]$

Step 1:

Find the critical points

$f'(x) = 3x^2 - 27$

$0 = 3x^2 - 27$

$= x^2 - 9$

$9 = x^2$

$x = \pm 3$

We reject $x = -3$ because it's not within the interval $[-1, 6]$.

$f(3) = (3)^3 - 27(3) + 1$

$= 27 - 81 + 1$

$= -53$

Step 2:

Find the $f(x)$ at the boundary points

$f(-1) = (-1)^3 - 27(-1) + 1$

$= -1 + 27 + 1$

$= 27$

$f(6) = (6)^3 - 27(6) + 1$

$= 216 - 162 + 1$

$= 55$

Step 3:

Abs max is at $x = 6$.

Abs min is at $x = 3$.

E.g. Find the abs max and abs min of f(x) in the interval
[-1, 6)

Step 1:

Look at previous page.

Step 2:

Since there is an open interval at $x=6$, we have to find the left sided limit at $x=6$.

$f(-1) = 27$

$\lim_{x \to 6^-} x^3 - 27x + 1$

$= \lim_{x \to 6^-} (6)^3 - 27(6) + 1$

$= 55$

Step 3

Abs min is at $x=3$

There is no abs max because f(6) has the highest value but its an open interval.

Side note:

When dealing with open intervals, you find the right limit for the smaller boundary and right limit for the larger boundary.

E.g. (a, b)

Find right hand limit

Find left hand limit

3. Rolle's Theorem:

If $f(x)$ is cont on $[a,b]$, diff on (a,b) and $f(a)=f(b)$, then there exists at least 1 number $c \in (a,b)$ such that $f'(c)=0$.

Proof.

If $f(x)$ is cont on $[a,b]$, diff on (a,b) and $f(a)=f(b)$, then by EVT, $f(x)$ has an abs max and an abs min between $[a,b]$.

- Case 1:

$f(x)=C$, C is a constant

Since $f'(x)=C$, then $f'(x)=0$ for all numbers in $[a,b]$.

- Case 2:

$f(x) \neq C$, and $f(x)$ attains one of the global extremum on the endpoints of $[a,b]$. Since $f(a)=f(b)$, then $f(x)$ attains local extremum at point c on open interval (a,b) and by Fermat's Theorem, $f'(c)=0$.

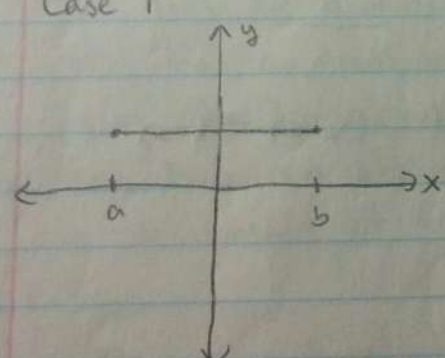
- Case 3:

$f(x) \neq C$ and $f(x)$ attains both global extrema at the interior points of $[a,b]$ then $f(x)$ attains its local extrema in the open interval (a,b) and by Fermat's theorem $f'(c)=0$.

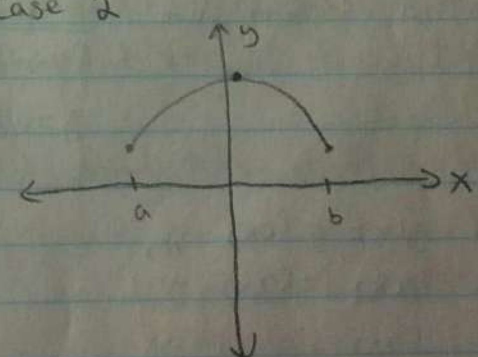
$\therefore$ Rolle's Theorem is proved for any function.

QED

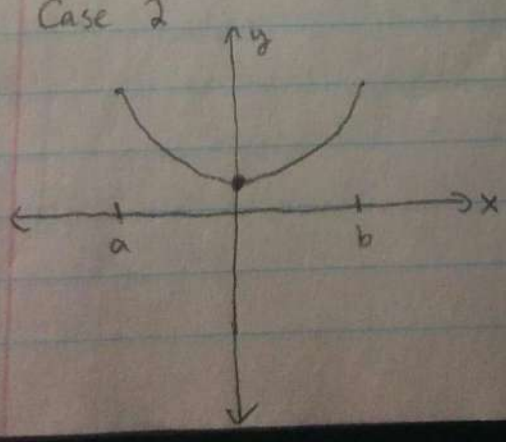
Case 1



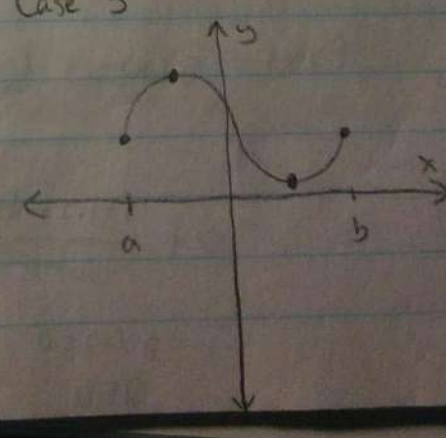
Case 2



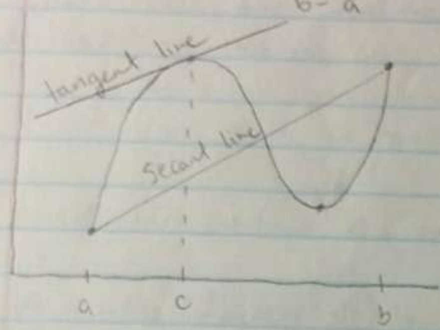
Case 2



Case 3



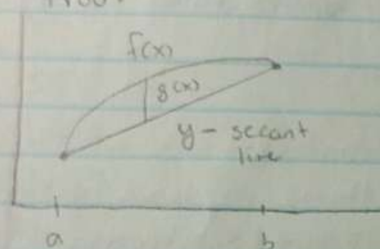
4. Lagrange's Theorem / Mean Value Theorem (MVT):
If $f(x)$ is cont on the closed interval $[a,b]$ and diff on open interval (a,b) then there exists at least 1 number $c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.



$\frac{f(b)-f(a)}{b-a}$ is average rate of change on $[a,b]$.

$f'(c)$ is instanteous rate of change at $c \in (a,b)$.

Proof



Let $g(x) = f(x) - y$

Since $g(x)$ is cont on $[a,b]$, $g(x)$ is diff on (a,b) and $g(a) = g(b)$, By Rolle's Theorem, $\exists\, c \in (a,b)$ such that $g'(c) = 0$

Eqn of secant line: $\frac{y - f(a)}{f(b)-f(a)} = \frac{x-a}{b-a}$

$$y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

Since $g(x) = f(x) - y$,

$$g'(x) = f'(x) - y'$$

$$f'(x) = g'(x) + y'$$

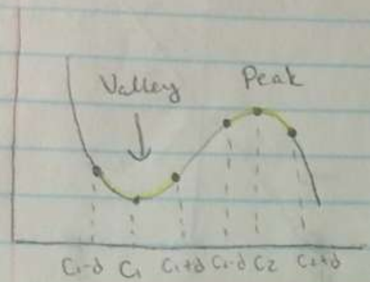
$$f'(x) = g'(x) + \frac{f(b)-f(a)}{b-a}$$

$$= 0 + \frac{f(b)-f(a)}{b-a}$$

$\therefore$ $g'(c) = 0$

QED

5. Local (Relative) Extrema
Let $f(x)$ be a function defined on open interval (a,b).
$f(x)$ has a local min at $x=c$ if there exists some $d>0$ such that $f(c) \le f(x)$ for all $x \in (c-d, c+d)$.
$f(x)$ has a local max at point $x=c$ if there exists some some $d>0$ such that $f(c) \ge f(x)$ for all $x \in (c-d, c+d)$.

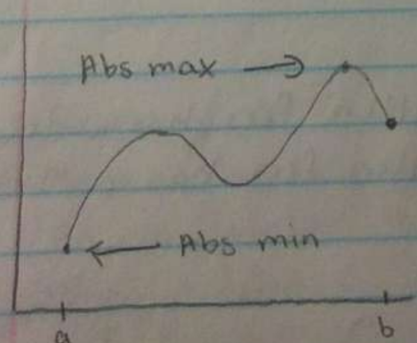


A local extrema can either be a peak or a valley.

To find local extrema, calculate $f'(x)=0$.

6. Critical Point:
If $x=c$ is in the domain of $f(x)$ such that either $f'(c)=0$ or $f'(c)=$ undefined, then that point is a critical point.
$f'(x)=0$ or $f'(x)=$ undefined will calculate critical point(s)

7. Global (Absolute) Extrema
Let $f(x)$ be a function defined on closed interval $[a,b]$.
$f(x)$ has an abs max at point $x=c$ if $f(c) \ge f(x)$ for all x in the domain of $f(x)$.
$f(x)$ has an abs min at point $x=c$ if $f(c) \le f(x)$ for all x in the domain of $f(x)$.



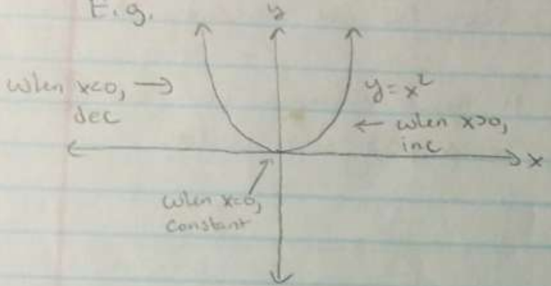
8. How derivatives affect the shape of a graph:

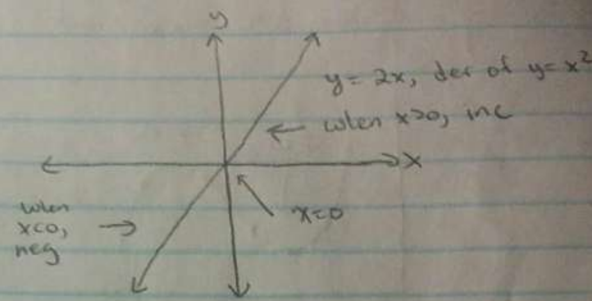
a) If $f'(x) > 0$ for all $x \in (a,b)$ then $f(x)$ is inc on (a,b)

b) If $f'(x) < 0$ for all $x \in (a,b)$ then $f(x)$ is dec on (a,b)

c) If $f'(x) = 0$ for all $x \in (a,b)$ then $f(x)$ is constant on (a,b)

E.g.





9. First Derivative Test

a) If $f'(x)$ changes from pos to neg at c, then $f(x)$ has a local max at c.

b) If $f'(x)$ changes from neg to pos at c, then $f(x)$ has a local min at c.

c) If $f'(x)$ doesn't change sign at c, then $f(x)$ has no max or min at c

10. Concavity Test

$f(x)$ concaves up if $f''(x) > 0$

$f(x)$ concaves down if $f''(x) < 0$

To find the inflection points, set $f''(x)$ to 0 and solve for x. An inflection point only exists iff $f''(x)$ changes its sign at the points you solved for x.

Second Derivative Test

a) If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ has a local max at c

b) If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ has a local min at c

11. Curve Sketching

E.g. Draw the graph of $f(x) = \frac{1}{x} + \frac{1}{x^2}$

Step 1. Find the Domain of $f(x)$

In this case, Dom $f(x)$: $x \in \mathbb{R} \mid x \neq 0$

Step 2. Find the x and y intercepts

To find x-int, sub $y=0$ and solve for x.

To find y-int, sub $x=0$ and solve for y.

In this case:

x-int:

$$0 = \frac{1}{x} + \frac{1}{x^2}$$

$$= \frac{x+1}{x^2}$$

$x = -1$

$(-1, 0)$

y-int:

There does not exist a y-int because x can't be 0.

DNE

Step 3. Find the symmetry of $f(x)$.

An even function exists if $f(-x) = f(x)$.

An odd function exists if $f(-x) = -f(x)$.

In this case, $f(-x) = \frac{1}{-x} + \frac{1}{x^2}$

$$f(x) = \frac{1}{x} + \frac{1}{x^2}$$

$$-f(x) = \frac{1}{x} - \frac{1}{x^2}$$

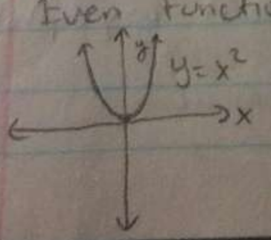
$\therefore$ Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, $f(x)$ is neither even nor odd.

Remainder *

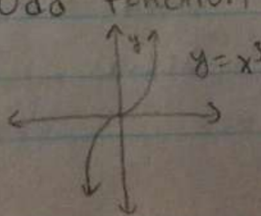
An even function has symmetry about the y-axis.

An odd function has symmetry about the origin.

Even Function



Odd Function



Step 4. Find asymptotes of $f(x)$.
There are 3 types of asymptotes, vertical (VA), horizontal (HA) and slant (SA).

Vertical Asymptotes:
To find VA, factor the denominator and see where $x=0$.

E.g. $f(x) = \frac{1}{x} + \frac{1}{x^2}$

In this case, VA would be at $x=0$.

Horizontal Asymptote:
To find HA compare the highest degree of the numerator with the highest degree of the denominator. If the highest degree of the numerator is less than the highest degree of the denominator, $y=0$ is the HA. If they are equal, HA = $\frac{\text{leading coefficient of num}}{\text{leading coefficient of deno}}$

E.g. $f(x) = \frac{3x^2 + 5x - 4}{4x^3 - 3x^2 + 8}$

Since $2 < 3$, HA: $y=0$

$f(x) = \frac{3x^4 + 5x - 4}{4x^4 + 5x - 8}$

Since $4=4$, HA: $y = \frac{3}{4}$

In this case, $f(x) = \frac{x+1}{x^2}$, so HA: $y=0$

Slant Asymptote:
Only occurs if the leading degree of the numerator is 1 more than the leading degree of the denominator.

E.g. $f(x) = \frac{5x^4 - 3x + 8}{3x^3 - 4x + 3}$

Since $4 = 3+1$, a slant asymptote occurs.

To find the slant asymptote, you divide the numerator by the denominator.

E.g. $f(x) = \dfrac{5x^4 - 3x + 8}{3x^3 - 4x + 3}$

$$\begin{array}{r|l} & \frac{5}{3}x \\ 3x^3-4x+3 & 5x^4 - 3x + 8 \\ & \underline{5x^4} \\ & -3x + 8 \end{array}$$

∴ The eqn of SA is $y = \frac{5}{3}x$

Side Notes *

A function can't cross VA but can cross HA and SA.
A function cannot have both HA and SA. It either has a HA or a SA or neither. To check if f(x) crosses HA or SA, solve f(x) = HA or f(x) = SA. The value(s) of x solved will intersect HA or SA.

Step 5. Find Critical Points

To find the critical points, solve $f'(x) = 0$. Then, to find if f(x) is increasing or decreasing,
↳ or $f'(x) = DNE$

In this case, $f'(x) = \dfrac{-1}{x^2} - \dfrac{2}{x^3}$

$$0 = \frac{-x-2}{x^3}$$
$$= -x - 2$$
$$= x + 2$$

$x = 0$ and $x = -2$ are the critical points

	$x < -2$	$-2 < x < 0$	$x > 0$
$-(x+2)$	+	−	−
x^3	−	−	+
Sign	−	+	−

↑ f(x) is dec when $x < -2$

↑ f(x) is inc when $-2 < x < 0$

↑ f(x) is dec when $x > 0$

Step 6. Concavity

To find the inflection points, solve $f''(x)=0$ or $f''(x)=DNE$.

To find where $f(x)$ is concaving up or down, draw a sign chart.

E.g. $f(x) = \frac{1}{x} + \frac{1}{x^2}$

$$f'(x) = \frac{-x-2}{x^3}$$

$$f''(x) = \frac{2x+6}{x^4}$$

Since x^4 is in the denominator, $x \neq 0$.

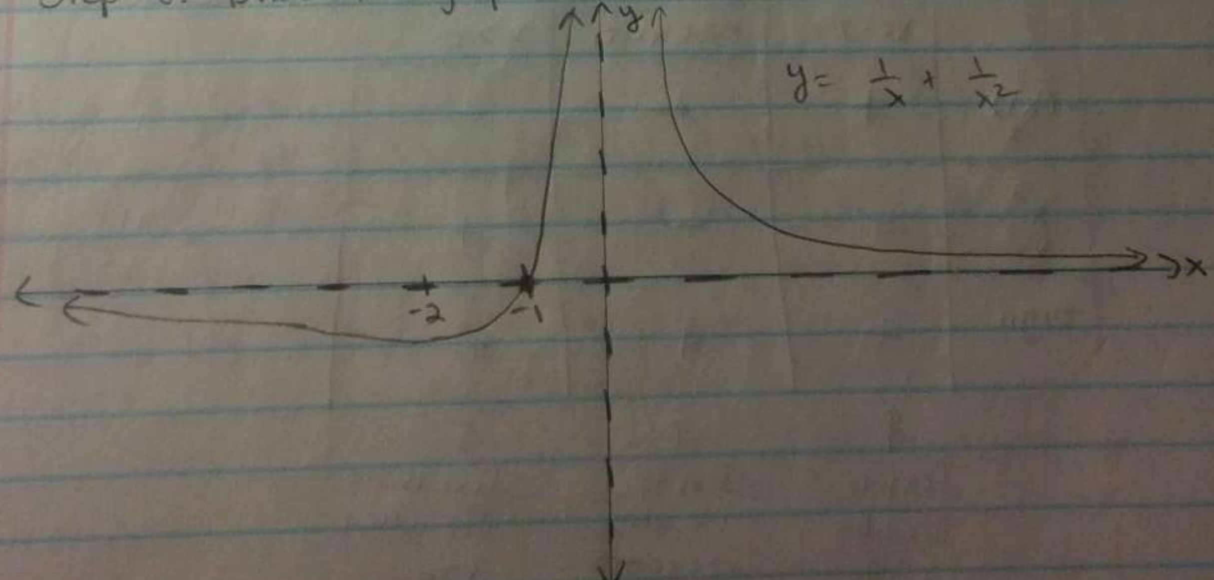
$0 = 2x+6$

$x = -3$

$\therefore$ $x=0$ and $x=-3$ are the inflection points

	$x<-3$	$-3<x<0$	$x>0$
$2x+6$	$-$	$+$	$+$
x^4	$+$	$+$	$+$
Sign	$-$	$+$	$+$
Concavity	$\cap$	$\cup$	$\cup$

Step 8. Draw the graph

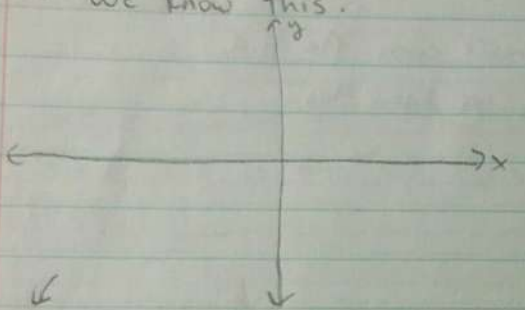


Side Note *

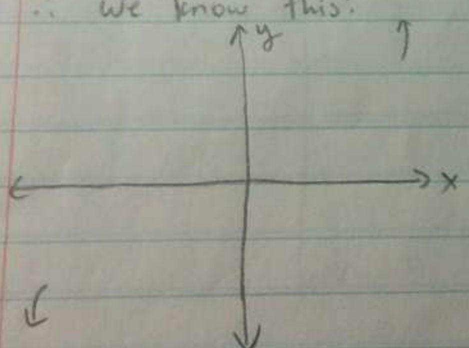
To see how your function behaves at the endpoints, sub in a large negative value for x and see if f(x) is positive or negative and a large positive value for x and see if f(x) is positive or negative.

E.g. Consider the function $f(x) = x^5 - x^4$.

1. Sub a large negative number for x.
   Let's say $x = -10$
   $f(-100) = -110{,}000$
   ∴ We know this:



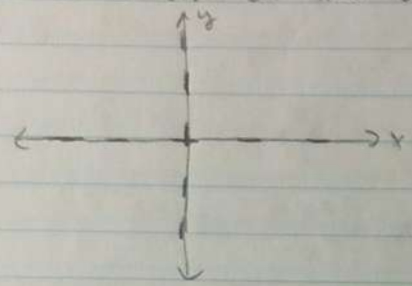
2. Sub in a large positive number for x.
   Say $x = 10$
   $f(100) = 90{,}000$
   ∴ we know this:



Suppose you have V.A., H.A. or S.A.

E.g. Consider the function $f(x) = \frac{x-1}{x}$

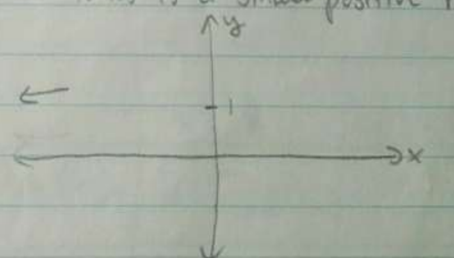
There is a V.A. at $x=0$ and a H.A. at $y=1$.



1. Sub in -10 for x

$f(-10) = 1.1 \rightarrow$ Since $f(x) > 1$, $f(x)$ doesn't cross the H.A.

Since $f(-10)$ is a small positive number, we know this:

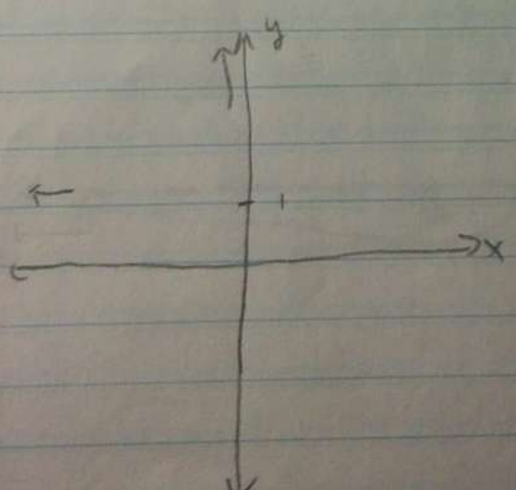


2. Sub in a value for x that is close to $x =$ V.A., but smaller

Since V.A. occurs when $x=0$, sub $x = -\frac{1}{2}$

$$f(-\tfrac{1}{2}) = \frac{-\frac{1}{2}-1}{-\frac{1}{2}}$$
$$= -2(-\tfrac{3}{2})$$
$$= 6$$

Since $f(-\frac{1}{2})$ is positive, we know this:

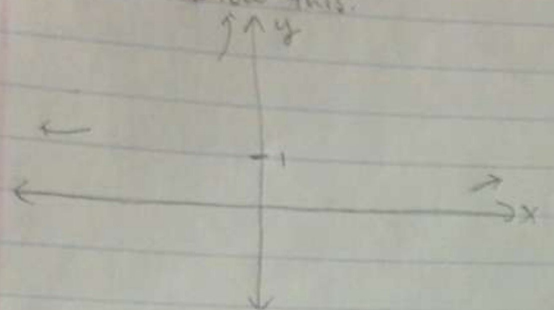


3. Sub in 10 for x

$f(10) = \frac{10-1}{10}$

$= 0.9 \rightarrow$ Since $f(x) < 1$, we know $f(x)$ doesn't cross the H.A.

$\therefore$ We know this:



4. Sub in a value for x that is close, but greater than the V.A.

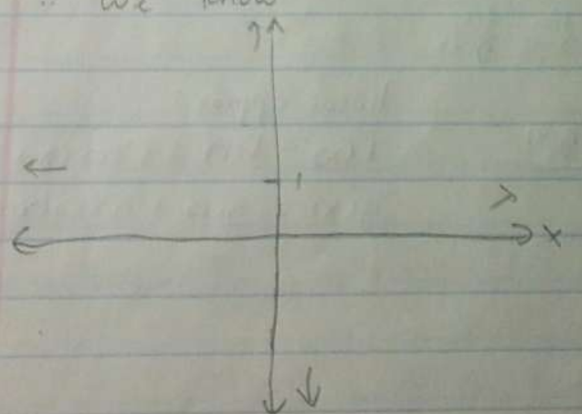
Since $x=0$ is the V.A., sub in $\frac{1}{2}$ for x

$f(\frac{1}{2}) = \frac{\frac{1}{2}-1}{\frac{1}{2}}$

$= 2(-\frac{1}{2})$

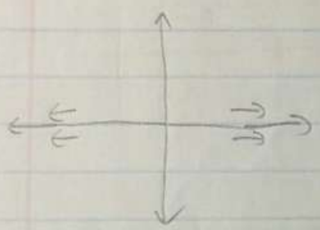
$= -1$

$\therefore$ We know

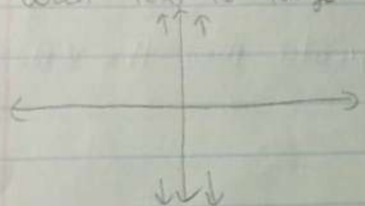


When you sub a large number for x, and f(x) is a small number, you know that f(x) will be approaching the H.A. Otherwise, if f(x) is a large number, then it will either go up or down.

When f(x) is small:



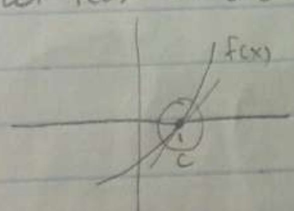
When f(x) is large:

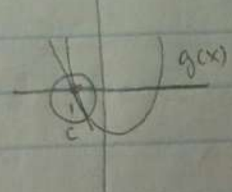


12. L'hopital's Rule

Proof:

Let $f(c) \to 0$ and $g(c) \to 0$ and $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$





linear approx:

$f(x) = f(c) + f'(c)(x-c)$

$g(x) = g(c) + g'(c)(x-c)$

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

$$= \lim_{x \to c} \frac{\overset{0}{\cancel{f(c)}} + f'(c)\cancel{(x-c)}}{\underset{0}{\cancel{g(c)}} + g'(c)\cancel{(x-c)}}$$

$$= \lim_{x \to c} \frac{f'(c)}{g'(c)}$$

L'hopital's Rule

If we have one of the following cases:

1. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$

2. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ c can be a real number or $\pm\infty$

we will do $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ to solve it.

E.g. $\lim_{x \to 0} \frac{\sin x}{x}$

Subbing $x=0$ will give us $\frac{0}{0}$.
$\therefore$ Using l'hopital's rule, we get $\lim_{x \to 0} \frac{\cos x}{1}$

$= 1.$

E.g. $\lim_{x \to \infty} \frac{e^x}{x^2}$

Subbing $x=0$ will give us $\frac{\infty}{\infty}$.
$\therefore$ Using l'hopital's rule, we get $\lim_{x \to \infty} \frac{e^x}{2x}$.

This still gives us $\frac{\infty}{\infty}$, so apply l'hopital's rule again.
This time, we get $\lim_{x \to \infty} \frac{e^x}{2}$

$= \infty$

L'hopital's rule can be used more than once.

E.g. $\lim_{x \to 0^+} x\ln(x)$

This is in the form of $0(\infty)$, which is still indeterminate, but can't directly be solved with l'hopital's rule.
To fix this, we do $\lim_{x \to 0^+} \frac{\ln x}{1/x}$. Now, we can

apply l'hopital's rule. $\lim_{x \to 0^+} \frac{\ln x}{1/x}$

$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$

$= \lim_{x \to 0^+} -x = 0$

To solve problems that have $(0)(\pm\infty)$, turn $f(x)\cdot g(x)$ into $\frac{f(x)}{1/g(x)}$ or $\frac{g(x)}{1/f(x)}$.

If you turned the product into a fraction and find that you're making no progress, turn it the other way.

Lastly, if $f(x)$ is in any of these forms:

1. 1^{∞}
2. 0^0
3. ∞^0

take the ln of both sides and solve it.

E.g. $\lim_{x\to\infty} x^{\frac{1}{x}}$

This is in the form of ∞^0, so we need to take the ln of both sides.

$$y = x^{\frac{1}{x}}$$

$$\ln(y) = \frac{1}{x}\ln(x)$$

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(x)}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= 0$$

However, in the beginning, I said $y = x^{\frac{1}{x}}$, not $\ln(y)$.
But, remember $e^{\ln y} = y$.

$$\therefore \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln y}$$

$$= e^0$$

$$= 1$$

$$\therefore \lim_{x\to\infty} x^{\frac{1}{x}} = 1$$

1. If $\lim_{x \to c} \ln(f(x)) = L$, then $\lim_{x \to c} f(x) = e^L$

2. If $\lim_{x \to c} \ln(f(x)) = \infty$, then $\lim_{x \to c} f(x) = \infty$

3. If $\lim_{x \to c} \ln(f(x)) = -\infty$, then $\lim_{x \to c} f(x) = 0$

This is because $\lim_{x \to c} \ln(f(x)) = \ln\left(\lim_{x \to c} f(x)\right)$

1. $\lim_{x \to c} \ln(f(x)) = L$

$\ln\left(\lim_{x \to c} f(x)\right) = L$

$e^{\ln\left(\lim_{x \to c} f(x)\right)} = e^L$

$\lim_{x \to c} f(x) = e^L$

2. $\lim_{x \to c} \ln(f(x)) = \infty$

$\ln\left(\lim_{x \to c} f(x)\right) = \infty$

$e^{\ln\left(\lim_{x \to c} f(x)\right)} = e^{\infty}$

$\lim_{x \to c} f(x) = e^{\infty}$

$= \infty$

3. $\lim_{x \to c} \ln(f(x)) = -\infty$

$\ln\left(\lim_{x \to c} f(x)\right) = -\infty$

$e^{\ln\left(\lim_{x \to c} f(x)\right)} = e^{-\infty}$

$\lim_{x \to c} f(x) = e^{-\infty}$

$= 0$

Limit Laws

1. $\lim_{x \to c} x = c$

Proof:

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \rightarrow |x-c| < \varepsilon$

Choose d to be $x-c$

$|x-c| < \varepsilon$

$d = \varepsilon$

QED

2. $\lim_{x \to c} k = k$, k is a constant

Proof:

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \rightarrow |k-k| < \varepsilon$

3. $\lim_{x \to c} mx+b = mc+b$

$\forall \varepsilon > 0, \exists d > 0 \mid 0 < |x-c| < d \rightarrow |mx+b-(mc+b)| < \varepsilon$

$|mx+b-(mc+b)|$

$= |mx+b-mc-b|$

$= |mx-mc|$

$= |m(x-c)|$

$= m|x-c|$

$= md$

$md = \varepsilon$

$d = \dfrac{\varepsilon}{m}$

$|mx+b-(mc+b)| < \varepsilon$

$|m(x-c)| < \varepsilon$

$m\left(\dfrac{\varepsilon}{m}\right) = \varepsilon$

QED

4. Sum and Difference Law

$$\lim_{x\to a} [f \pm g] = \lim_{x\to a} f \pm \lim_{x\to a} g$$

5. Constant Multiplier Law

$$\lim_{x\to a} cf = c \lim_{x\to a} f$$

6. Product Law

$$\lim_{x\to a} [fg] = \left(\lim_{x\to a} f\right)\left(\lim_{x\to a} g\right)$$

7. Quotient Law

$$\lim_{x\to a} \left[\frac{f}{g}\right] = \frac{\lim_{x\to a} f}{\lim_{x\to a} g}, \quad \lim_{x\to a} g \neq 0$$

8. Power Law

$$\lim_{x\to a} (f)^n = \left(\lim_{x\to a} f\right)^n$$

9. Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ and $\lim_{x\to a} h(x) = L$ and $\lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f(x) = L$$

Math Proofs and Laws

Log Proofs

1. Product Rule

$\log_b(x \cdot y) = \log_b x + \log_b y$

Proof:

Let x be b^x.

Let y be b^y.

$LS = \log_b(b^x \cdot b^y)$

$= \log_b(b^{x+y})$

$= x+y$

$RS = \log_b(b^x) + \log_b(b^y)$

$= x+y$

$LS = RS$

QED

2. Quotient Rule

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

Proof:

Let x be b^x

Let y be b^y

$LS = \log_b\left(\frac{b^x}{b^y}\right)$

$= \log_b(b^{(x-y)})$

$= x-y$

$RS = \log_b(b^x) - \log_b(b^y)$

$= x-y$

$LS = RS$

QED

3. Power Rule

$\log_b x^y = y(\log_b x)$

Proof:

let x be b^x

$LS = \log_b (b^{x^y})$

$= \log_b (b^{yx})$

$= yx$

$RS = y(\log_b (b^x))$

$= yx$

$LS = RS$

QED

4. Change of Base Rule

$$\log_b x = \frac{\log_c x}{\log_c b}$$

Proof:

let $x = b^x$

$LS = \log_b (b^x)$

$= x$

$$RS = \frac{\log_c (b^x)}{\log_c b}$$

$$= \frac{x(\log_c b)}{\log_c b}$$

$= x$

$LS = RS$

QED

Summary of Log Laws

1. Product Law

$\log_b(xy) = \log_b x + \log_b y$

2. Quotient Law

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

3. Power Law

$\log_b(x^y) = y(\log_b x)$

4. Change of Base Law

$\log_b x = \frac{\log_c x}{\log_c b}$ A special case of this rule: $\log_b x = \frac{\log_x x}{\log_x b} = \frac{1}{\log_x b}$

5. Log Base Law

$\log_b b = 1$

6. Log of 1 Law

$\log_b 1 = 0$

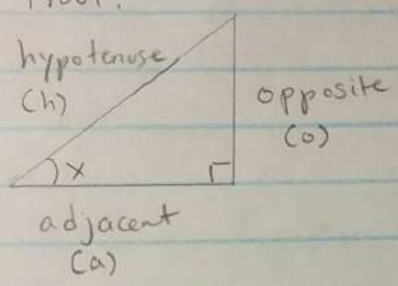
7. Restrictions

In $\log_b x$, x must be greater than 0. $\log_b x$ will be undefined if $x \leq 0$.
Furthermore, if $0 < x < 1$, $\log_b x$ will be negative.

Trig Proofs and Laws

1. $\cos^2 x + \sin^2 x = 1$

Proof:



$\sin x = \frac{o}{h}$ $\quad$ $\cos x = \frac{a}{h}$

$$
\begin{aligned}
LS &= \sin^2 x + \cos^2 x \\
&= \left(\frac{o}{h}\right)^2 + \left(\frac{a}{h}\right)^2 \\
&= \frac{o^2 + a^2}{h^2} \\
&= \frac{h^2}{h^2} \\
&= 1
\end{aligned}
$$

$LS = RS$

QED

2. $\tan^2 x + 1 = \sec^2 x$

Proof:

$\sin^2 x + \cos^2 x = 1$

If we divide both sides by $\cos^2 x$, we get

$\tan^2 x + 1 = \sec^2 x$

QED

3 $\cot^2 x + 1 = \csc^2 x$

Proof:

$\sin^2 x + \cos^2 x = 1$

If we divide both sides by $\sin^2 x$, we get

$\cot^2 x + 1 = \csc^2 x$

QED

4. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

Proof:

1. $\sin(x) = \cos(x - \frac{\pi}{2})$
2. $-\cos(x) = \sin(x - \frac{\pi}{2})$
3. $-\sin(x) = \cos(x + \frac{\pi}{2})$
4. $\cos(x) = \sin(x + \frac{\pi}{2})$

$$
\begin{aligned}
\sin(x+y) &= \cos(x+y-\tfrac{\pi}{2}) \\
&= \cos(x + (y - \tfrac{\pi}{2})) \\
&= \cos x \cos(y - \tfrac{\pi}{2}) - \sin x \sin(y - \tfrac{\pi}{2}) \\
&= \cos x \sin y - \sin x(-\cos y) \\
&= \sin x \cos y + \cos x \sin y
\end{aligned}
$$

QED

$$
\begin{aligned}
\sin(x-y) &= \cos(x-y-\tfrac{\pi}{2}) \\
&= \cos(x - (y + \tfrac{\pi}{2})) \\
&= \cos x \cos(y + \tfrac{\pi}{2}) + \sin x \sin(y + \tfrac{\pi}{2}) \\
&= \cos x(-\sin y) + \sin x \cos y \\
&= \sin x \cos y - \cos x \sin y
\end{aligned}
$$

QED

5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$$
\begin{aligned}
\cos(x+y) &= \sin(x+y+\tfrac{\pi}{2}) \\
&= \sin(x+(y+\tfrac{\pi}{2})) \\
&= \sin x \cos(y+\tfrac{\pi}{2}) + \sin(y+\tfrac{\pi}{2}) \cos x \\
&= \sin x(-\sin y) + \cos x \cos y \\
&= \cos x \cos y - \sin x \sin y
\end{aligned}
$$

QED

$$
\begin{aligned}
\cos(x-y) &= \sin(x-y+\tfrac{\pi}{2}) \\
&= \sin(x-(y-\tfrac{\pi}{2})) \\
&= \sin x \cos(y-\tfrac{\pi}{2}) - \cos x \sin(y-\tfrac{\pi}{2}) \\
&= \sin x(\sin y) - \cos x(-\cos y) \\
&= \cos x \cos y + \sin x \sin y
\end{aligned}
$$

QED

6. $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

Proof:

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)}$$

$$= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} \quad \begin{matrix}\text{Divide by } \cos x \cos y \\ \text{Divide by } \cos x \cos y\end{matrix}$$

$$= \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$$

QED

$$\tan(x-y) = \frac{\sin(x-y)}{\cos(x-y)}$$

$$= \frac{\sin x \cos y - \sin y \cos x}{\cos x \cos y + \sin x \sin y} \quad \begin{matrix}\text{(Divide by } \cos x \cos y) \\ \text{(Divide by } \cos x \cos y)\end{matrix}$$

$$= \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

QED

7. $\sin(2x) = 2\sin x \cos x$

Proof:

$\sin(2x) = \sin(x+x)$

$= \sin x \cos x + \sin x \cos x$

$= 2\sin x \cos x$

QED

8. $\cos(2x) = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

Proof:

$\cos(2x) = \cos(x+x)$

$= \cos x \cos x - \sin x \sin x$

$= \cos^2 x - \sin^2 x$

$= \cos^2 x - (1 - \cos^2 x)$ OR $= (1-\sin^2 x) - \sin^2 x$

$= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$

QED

9. $\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$

Proof:

$\tan(2x) = \dfrac{\sin(2x)}{\cos(2x)}$

$= \dfrac{2\sin x \cos x}{\cos^2 x - \sin^2 x}$ (Divide by $\cos^2 x$) (Divide by $\cos^2 x$)

$= \dfrac{2\tan x}{1-\tan^2 x}$

QED

10. $\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos x}{2}$

Proof:

We know that $\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) = 1$

$$\sin^2\left(\frac{x}{2}\right) = 1 - \cos^2\left(\frac{x}{2}\right)$$

$$= 1 - \left(\frac{1+\cos x}{2}\right)$$

$$= \frac{2 - 1 - \cos x}{2}$$

$$= \frac{1-\cos x}{2}$$

QED

11. $\cos^2\left(\frac{x}{2}\right) = \frac{1+\cos x}{2}$

Proof:

$$\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) = 1$$

$$\cos^2\left(\frac{x}{2}\right) = 1 - \sin^2\left(\frac{x}{2}\right)$$

$$= 1 - \left(\frac{1-\cos x}{2}\right)$$

$$= \frac{2 - 1 + \cos x}{2}$$

$$= \frac{1+\cos x}{2}$$

QED

Summary of Trig Laws

1. $\cos^2 x + \sin^2 x = 1$
2. $\tan^2 x + 1 = \sec^2 x$
3. $\cot^2 x + 1 = \csc^2 x$
4. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
5. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
6. $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
7. $\sin(2x) = 2\sin x \cos x$
8. $\cos(2x) = \cos^2 x - \sin^2 x$
   $= 2\cos^2 x - 1$
   $= 1 - 2\sin^2 x$
9. $\tan(2x) = \dfrac{2\tan x}{1 - \tan^2 x}$
10. $\sin^2\left(\frac{x}{2}\right) = \dfrac{1 - \cos x}{2}$
11. $\cos^2\left(\frac{x}{2}\right) = \dfrac{1 + \cos x}{2}$
12. $\sin(x) = \cos(x - \frac{\pi}{2})$
13. $-\cos(x) = \sin(x - \frac{\pi}{2})$
14. $-\sin(x) = \cos(x + \frac{\pi}{2})$
15. $\cos(x) = \sin(x + \frac{\pi}{2})$
16. $\sin(-x) = -\sin(x)$ (odd Function)
17. $\cos(-x) = \cos x$ (Even Function)
18. $\tan(-x) = -\tan x$ (odd Function)

Principal Values

1. $\sin x$: $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
2. $\cos x$: $x \in [0, \pi]$
3. $\tan x$: $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$
4. $\csc x$: $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $x \neq \pi$
5. $\sec x$: $x \in [0, \pi]$, $x \neq \frac{\pi}{2}$
6. $\cot x$: $x \in (0, \pi)$